Math 1050 : College Algebra
University of Utah

Chee Han Tan

Last modified : June 26, 2018

Contents

Preface

These notes are largely based on the textbook **Precalculus, 9th Edition, 2013 Larson**. These notes are by no means accurate, and any mistakes here are of course my own. Please report any typographical errors or mathematical fallacy to me by email tan@math.utah.edu.

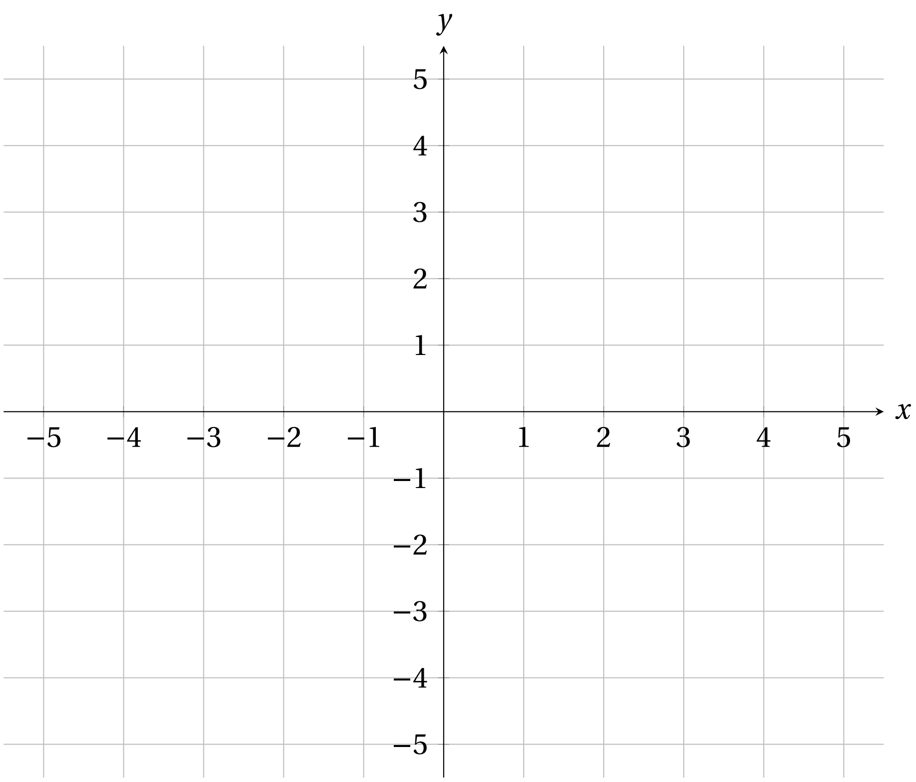
1 Functions and Their Graphs

1.1 Rectangular Coordinates

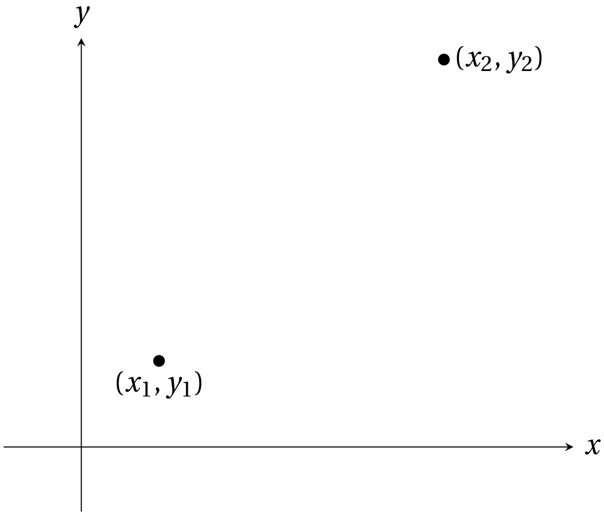
Real numbers are represented as points on the *real line*, but this is inadequate as soon as we consider pairs of real numbers, say $(2,3)$. Well, there are two real numbers so let us take two real lines and intersect them at right angle (90°), and this gives the **Cartesian plane**. One might question the logic behind intersecting two real lines to form the Cartesian plane. I think it is because we can become best friend with the *Pythagorean theorem*!

Definition 1.1. A *rectangular coordinate system* represents ordered pairs of real numbers as points on the Cartesian plane.

1. The horizontal line is called the ***x*-axis**. The vertical line is called the ***y*-axis**.
2. The point of intersection of the x and y-axes is called the **origin**, (0,0).
3. The 2 axes divide the Cartesian plane into 4 **quadrants**.
4. Each point in the plane corresponds to an ordered pair (x, y) of real numbers.



Any two points $(x_1, y_1), (x_2, y_2)$ on the Cartesian plane can be join by a straight line. There are two useful geometrical quantities.



- ♣ **Distance** between the points (that is, the length of the line), which can be found using the **Pythagorean theorem**.

$$\boxed{\text{Distance} = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

- ♣ **Midpoint** of the line, which is found by taking the **average** of the x-coordinates and the y-coordinates.

$$\boxed{\text{Midpoint} = M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)}$$

Example 1.2. For each of the given set of points, find the distance between them and the midpoint of the line segment joining them.

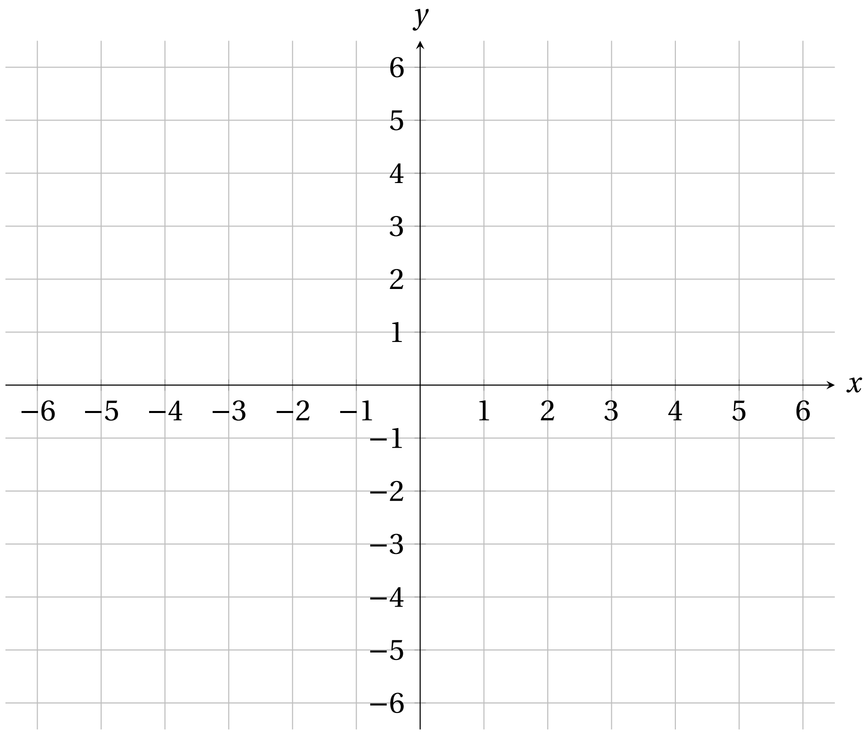
(a) $(-3, -5)$ and $(5, 1)$

(b) $(-4, 1)$ and $(3, -2)$

Example 1.3. Plot the following points on the Cartesian plane.

$$A = (-4, 1),\;\; B = (3, 6),\;\; C = (1, -5),\;\; D = (-5, -3),\;\; E = (5, 0),\;\; F = (0, -4).$$

Find the distance between A and C. Find the midpoint of the line segment joining B and D.



Note that $\sqrt{a+b}$ is not equal to $\sqrt{a} + \sqrt{b}$. Take $a = 9$ and $b = 16$. Then $\sqrt{a+b} = \sqrt{25} = 5$ but $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$.

1.2 Graphs of Equations

Knowing how to graph a given equation of two variables x and y is crucial in this course. Let us define **solutions** of an equation to be all ordered pairs (x, y) that satisfies the given equation upon substitution. The set of all points that are solutions of the equation then gives rise to the **graph** of the equation. A relatively easy method in graphing equation is by constructing a "T-table", where we pick some x-values and find its corresponding y-values, plot these points and make an educated guess about the graph.

Example 1.4. Consider the equation $y = x + 3$. Show that $(2, 5)$ is a solution but $(-3, 1)$ is not. Sketch the graph of the equation on the Cartesian plane.

Example 1.5. Consider the equation $y = x^2 + 1$. Show that $(-1, 2)$ is a solution but $(2, 9)$ is not. Sketch the graph of the equation on the Cartesian plane.

Intercepts of a graph

It is useful to identify points where the graph intersects the x- and y-axes. Let us name these special points since we will encounter them almost everywhere in the first three chapters.

- ***x*-intercepts** are points where the graph of an equation intersect the x-axis.
    - ♣ To find x-intercepts, we set ______________ and solve for __________.
- ***y*-intercepts** are points where the graph of an equation intersect the y-axis.
    - ♣ To find y-intercepts, we set ______________ and solve for __________.

Example 1.6. Find the x- and y-intercepts of the graph of the following equations.

(a) $y = 5x - 6$

(b) $y = 7 + \sqrt{x + 4}$

(c) $y = \sqrt{x - 1} - 3$

Symmetry

Symmetries are ubiquitous in nature and aside from the aesthetic appeal, they are quite useful in graphing equations because we can avoid sketching the whole graph. Clearly, symmetries are easily detected if the graph of an equation is given (duhhhh), but this is the opposite of what we wanted: exploiting symmetries to sketch the graph. *Wouldn't it be great if we have a way of checking symmetries without having to graph the equation?*

Algebraic Tests for Symmetry: Given an equation of two variables, its graph is

1. **symmetric with respect to the x-axis** (horizontal reflection) when replacing __________________________ yields the same equation;
2. **symmetric with respect to the y-axis** (vertical reflection) when replacing __________________________ yields the same equation;
3. **symmetric with respect to the origin** (both horizontal and vertical reflection) when replacing __________________________ yields the same equation.

Example 1.7. For the following equations, use the algebraic test to check for symmetry with respect to both axes and the origin.

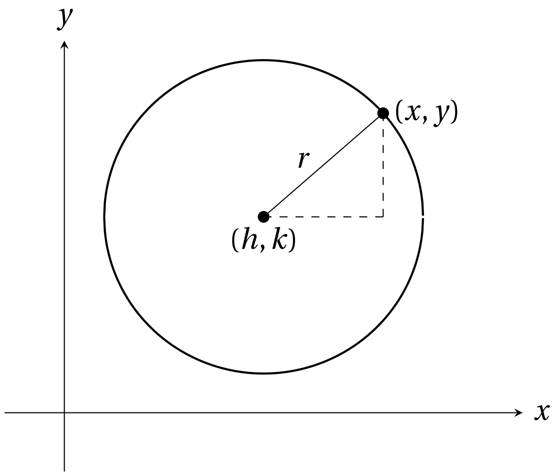
(a) $x = y^2 - 1$

(b) $y^2 = \dfrac{1}{x^2 + 3}$

(c) $y = x^3 - 3x$

Circles

It seems reasonable to think that graphs with beautiful symmetries can be represented with a single equation. This is actually very difficult (try equilateral triangles or rectangles or peanut-shaped if you do not believe me), with a few exceptions such as circles.



A **circle** with **center** (h, k) and **radius** $r > 0$ is all points such that the distance from the center (h, k) is r. Let (x, y) be any point on the circle. It follows from the Pythagorean theorem that (x, y) satisfies the following equation, which is called the **standard from of the equation of the circle**.

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

Example 1.8. Write the standard form of the equation of the circle with

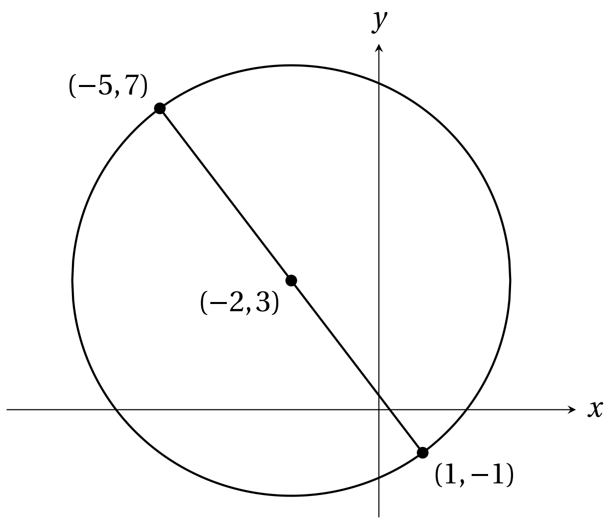
(a) center $(5, 8)$ and radius 3;

(b) center $(-3, 2)$ and radius 7.

Example 1.9. The point $(3, 1)$ lies on a circle whose center is at $(-1, 2)$. Find the equation of this circle in standard form.

Example 1.10. Write the equation of the circle that has endpoints of its diameter at $(-5,7)$ and $(1,-1)$. Find the x- and y-intercepts, if any.

Solution:



We sketch the given points and the line segment joining them to better visualise the circle. The center of the circle is the midpoint M of this line segment:

$$M = \left(\frac{-5+1}{2}, \frac{7-1}{2}\right) = (-2,3) = (h,k).$$

The radius of the circle is defined to be the distance between the center and any point on the circle. Choosing the point $(1,-1)$, the radius is

$$\begin{aligned} r &= \sqrt{(1-(-2))^2 + (-1-3)^2} \\ &= \sqrt{(3)^2 + (-4)^2} = 5. \end{aligned}$$

Hence, the equation of the circle in standard form is

$$(x-(-2))^2 + (y-3)^2 = 5^2 \quad \text{or} \quad (x+2)^2 + (y-3)^2 = 25.$$

To find the x-intercepts, we set $y = 0$ and solve for x:

$$\begin{aligned} (x+2)^2 + (-3)^2 &= 25 \\ (x+2)^2 + 9 &= 25 \\ (x+2)^2 &= 16 && \left[\text{Substract 9 from each side}\right] \\ x+2 &= \pm 4 && \left[\text{Take the square root of each side}\right] \\ x &= -2 \pm 4. && \left[\text{Subtract 2 from each side}\right] \end{aligned}$$

Thus the x-intercepts are $(-2-4,0) = (-6,0)$ and $(-2+4,0) = (2,0)$. To find the y-intercepts, we set $x = 0$ and solve for y:

$$\begin{aligned} (2)^2 + (y-3)^2 &= 25 \\ 4 + (y-3)^2 &= 25 \\ (y-3)^2 &= 21 && \left[\text{Substract 4 from each side}\right] \\ y-3 &= \pm\sqrt{21} && \left[\text{Take the square root of each side}\right] \\ y &= 3 \pm \sqrt{21}. && \left[\text{Add 3 to each side}\right] \end{aligned}$$

Thus the y-intercepts are $(3-\sqrt{21},0)$ and $(3+\sqrt{21},0)$.

1.3 Linear Equations in Two Variables

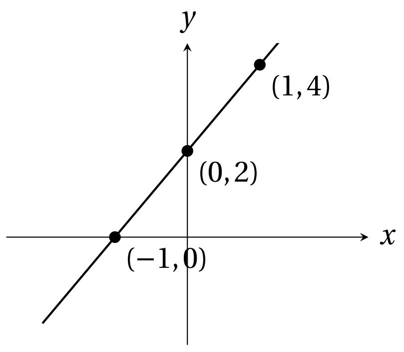
The simplest model that relates two variables is the *linear equation in two variables* and its graph is a straight line. *Just how much information is required to write down an equation of a given line?*

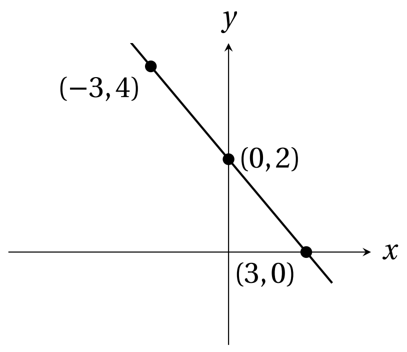
Slope of a line

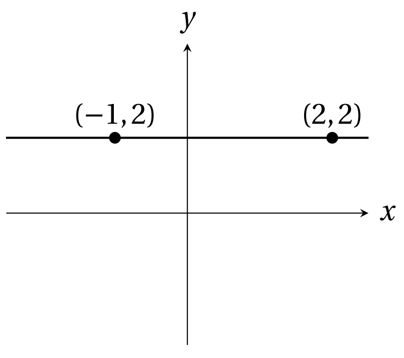
It seems like we need a notion of the "steepness" of a line. Here is the long-awaited definition of a slope. Given a non-vertical line passing through (x_1, y_1) and (x_2, y_2), its **slope** m is defined as follows.

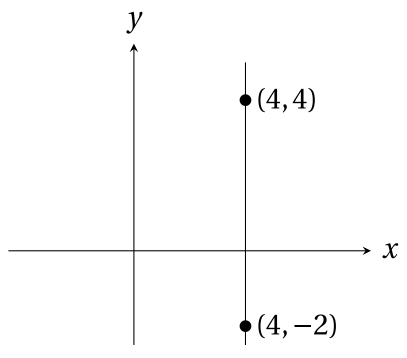
$$\boxed{\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise/fall}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}}$$

Below we demonstrate that slope can be positive, negative, zero or undefined. But the takeaway here is that **the slope of a line determines the shape of the line**: the line rises if $m > 0$ and falls if $m < 0$.









There are two forms of equation for a given line.

$$\boxed{\begin{aligned} \textbf{Point-slope form}: \quad & y - y_1 = m(x - x_1) \\ \textbf{Slope-intercept form}: \quad & y = mx + b \end{aligned}}$$

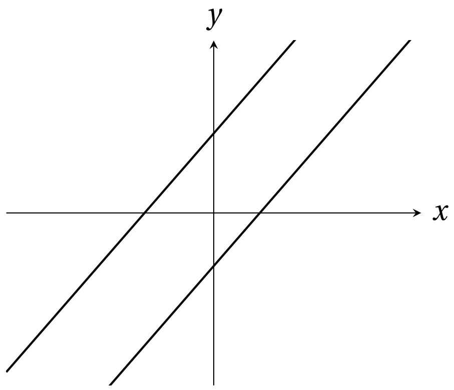
To find an equation of a given line, we need to know the slope and a point on the line.

Example 1.11 (A slope and a point on the line are given)**.** Find an equation of the line that has a slope of 2 and passes through the point $(3, -1)$. Find the intercepts.

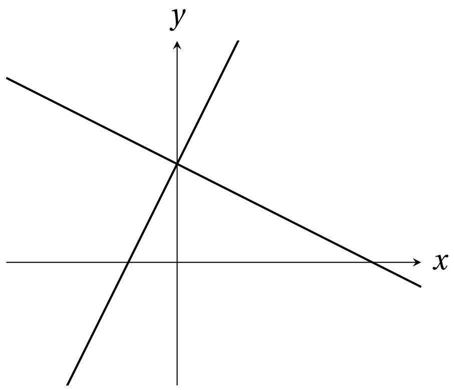
Example 1.12 (Two points on the line are given)**.** Find an equation of the line that passes through the points $(4, -2)$ and $(2, 5)$. Find the intercepts.

It is common to mess up the order of coordinate values when computing the slope. For this reason, I strongly encourage you to sketch the given points and line and check that the slope matches the shape of the line.

Parallel and perpendicular lines



They either overlap and thus are the exact same line, or they do not overlap but they never intersect each other. Such lines are called **parallel lines** and they share the same slope $m_1 = m_2$.



They intersect at a right angle (90°). Such lines are called **perpendicular lines** and their slopes satisfies the following relation

$$m_1 m_2 = -1.$$

That is, the slopes are negative reciprocal of each other.

Example 1.13. Find an equation of the line that passes through the point $(3,-1)$ and are

(a) parallel to the line $2x - 3y = -1$;

(b) perpendicular to the line $3y + 4x = 2$.

Example 1.14. In-state tuition at the University of Orange in 2014 was \$6889 and \$7203 in 2016. Assuming the tuition started growing linearly in 2012, write a function describing the tuition and use it to determine how much tuition it was in 2011.

Solution: Some of you probably never hear about the concept of a function before this course, but hopefully it is clear that function in this example refers to (linear) equation. Since the tuition grows linearly by assumption, let $y = F(x)$ be the linear function describing the tuition at time x in years, where the year 2014 corresponds to $x = 0$. This means that the year 2016 corresponds to $x = 2$. To this end, let $(x_1, y_1) = (0, 6889)$ and $(x_2, y_2) = (2, 7203)$. The slope of this line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7203 - 6889}{2 - 0} = \frac{314}{2} = 157,$$

which matches the description in the question that the tuition is growing. Since $(0, 6889)$ is the y-intercept of this line, the equation of $y = F(x)$, in slope-intercept form, is

$$y = mx + b = 157x + 6889 = F(x).$$

Finally, the tuition in year 2011, corresponding to time $x = -3$, is

$$y = 157(-3) + 6889 = \$6418.$$

1.4 Functions

A **function** f from a set A to a set B is a relation that assigns every element x in A to exactly one element y in B. The set A is the **domain** (or set of inputs) and the set B contains the **range** (or set of outputs).

1. Each element in A must be matched with an element in B.
2. Some elements in B might not be matched with any element in A.
3. Two or more elements in A may be matched with the same element in B, but an element in A can only matched with exactly one element in B.

Example 1.15. Determine which of the following equations represent y as a function of x.

(a) $x^2 + y^2 = 4$

(b) $2x + 3y = 4$

(c) $y = \sqrt{16 - x^2}$

(d) $|y| = 4 - x$

Example 1.16. Consider the function $f(x) = 3x + \sqrt{x+1}$. **Evaluating a function at a given value** simply means replacing the independent variable or input with the given value. For instance,

$$f(1) = 3(1) + \sqrt{1+1} = 3 + \sqrt{2}.$$

Evaluate $f(0), f(-2), f(x+2), f(3-t)$.

Piecewise-defined functions

Piecewise-defined functions are functions defined by two or more equations called subfunctions, each over a specified domain. Consider the following piecewise-defined function:

$$f(x) = \begin{cases} -2x+4 & \text{if } x < 0, \\ x+5 & \text{if } x \geq 1. \end{cases}$$

which has 2 subfunctions. To evaluate f for a specific x-value, we need to **decide which domain it belongs to** and then **substitute into that particular subfunction**.

What is $f(-2) =$ ____________________? What is $f(1) =$ ____________________?

What is $f(0) =$ ____________________? What is $f(3) =$ ____________________?

To sketch the graph of $f(x)$ above, we sketch each subfunction over its domain and then combine them altogether into a single graph. To this end, we introduce the notion of **open circle** and **closed circle**.

Example 1.17. Given $f(x) = x^2 - x + 7$ and $g(x) = 19 - 5x$, find all values of x for which $f(x) = g(x)$.

Algebraically determine the domain of a function

Graphically, we determine the domain of a function by "scanning" the graph from left to right and look for any "gaps" in the graph. Given a function f, if the domain is not specified, then the **implied domain** is the set of all real numbers for which the expression is defined. This excludes two situations:

1. Division by zero.
2. Taking an even root of a negative real number.

Example 1.18. Find the domain of the following functions.

(a) $f(x) = -5x + 8$

(b) $f(x) = 2(x-3)^2$

(c) $f(x) = \dfrac{2}{x-5}$

(d) $f(x) = \sqrt{4x+3}$

(e) $f(x) = |5x - 3|$

(f) $f(x) = \dfrac{x+3}{9-5x}$

1.5 Analyzing Graphs of Functions

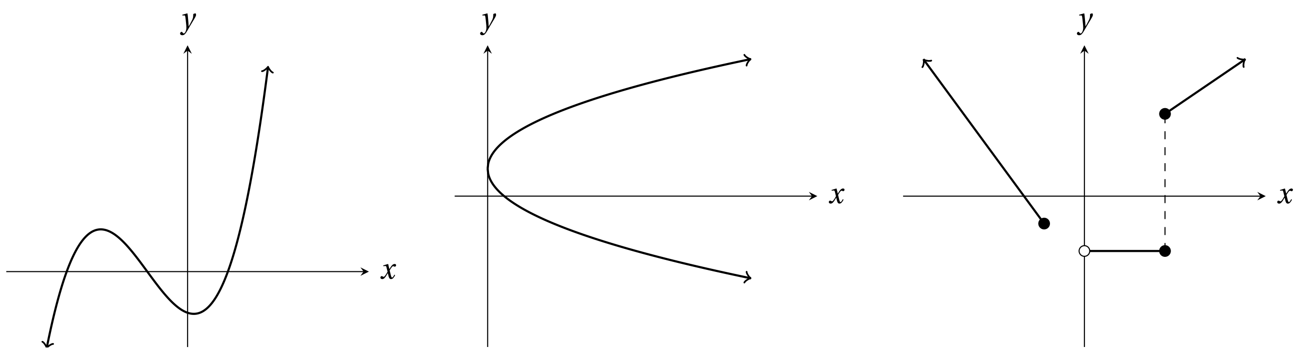
We begin by stating the **Vertical Line Test** for functions:

A given graph in a plane is the graph of y as a function of x

$\Updownarrow$

Any vertical line intersects the graph at most one point.

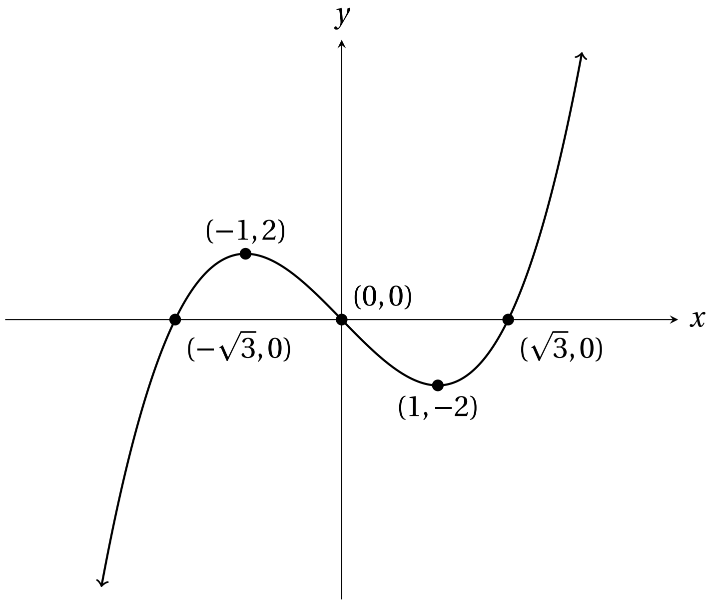
Example 1.19. Use the Vertical Line Test to determine which each of the following graphs represents y as a function of x.



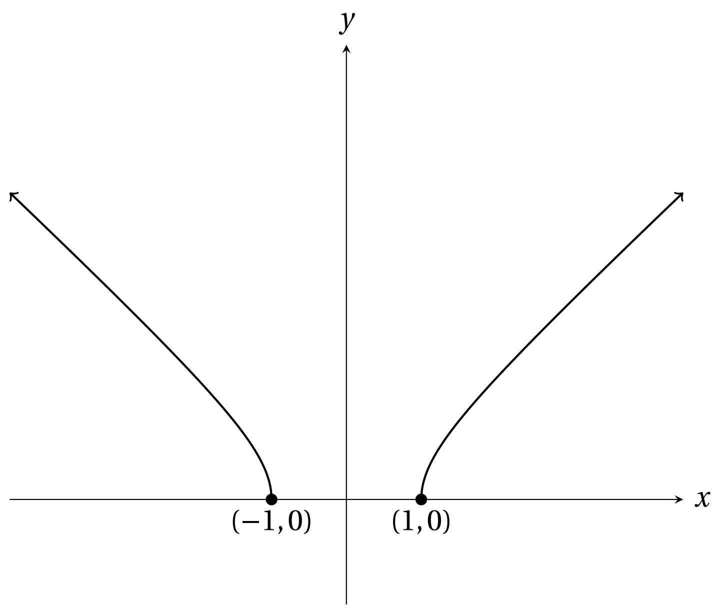
The goal in this section is to learn how to extract important information from a given graph.

1. **Domain and range**
   - For domain, look for gaps from left to right of the graph.
   - For range, look for gaps from bottom to top of the graph.
   - Be careful of open and closed circles.
2. **Zeros of a function**
   - The zeros (or roots) of a function $f(x)$ are all x-values for which $f(x) = 0$.
   - To find zeros of $f(x)$, set $f(x) = 0$ and solve for x.
   - This is equivalent to finding the x-intercepts of the graph.
3. **Increasing and decreasing intervals**
   - These must be open intervals.
4. **Relative minimum and maximum**
5. **Symmetries**
   - A function $y = f(x)$ is **odd** if its graph is symmetric with respect to the origin. Algebraically this means $f(-x) = -f(x)$ for all x in the domain of f.
   - A function $y = f(x))$ is **even** if its graph is symmetric with respect to the y-axis. Algebraically this means $f(-x) = f(x)$ for all x in the domain of f.

Example 1.20. Find the domain, range, zeros, increasing and decreasing intervals and relative minimum and maximum of the function $y = f(x) = x^3 - 3x$. Determine whether $f(x)$ is even or odd.

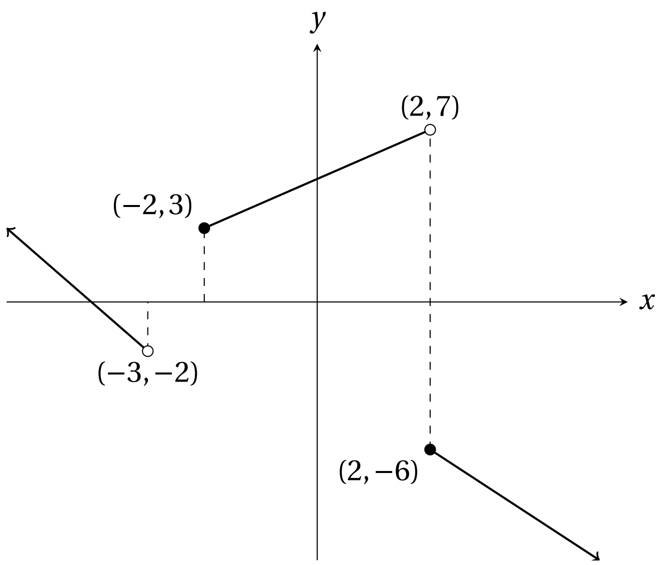


Example 1.21. Find the domain, range, zeros, increasing and decreasing intervals and relative minimum and maximum of the function $y = f(x) = \sqrt{x^2 - 1}$. Determine whether $f(x)$ is even or odd.



Example 1.22. Find the domain, range, zeros, increasing and decreasing intervals of $f(x)$.

$$y = f(x) = \begin{cases} -2x-8 & \text{if } x < -3, \\ x+5 & \text{if } -2 \le x < 2, \\ -1.5x-3 & \text{if } x \ge 2. \end{cases}$$



Domain: Scanning the graph from left to right, we see that the graph

- starts at $x = -\infty$ and stops at $x = -3$;
- starts again at $x = -2$ and stops at $x = 2$;
- starts again at $x = 2$ and ends at $x = +\infty$.

Since we have an open circle at $x = -3, 2$ and closed circles at $x = -2, 2$, the domain is $(-\infty, -3) \cup [-2, 2) \cup [2, +\infty]$.

Range: Scanning the graph from bottom to top, we see that the graph

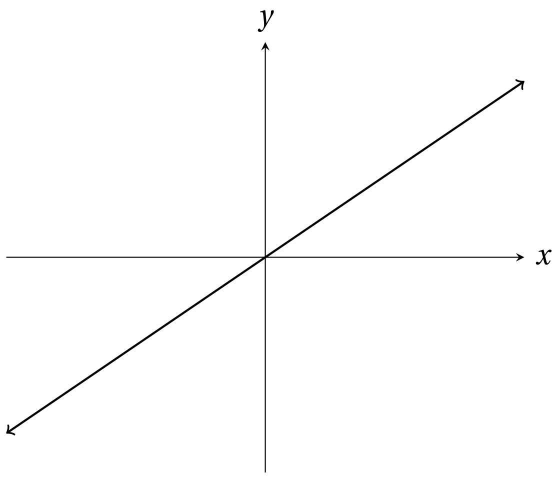
- starts at $y = -\infty$ and stops at $y = -6$;
- starts again at $y = -2$ and finally ends at $y = +\infty$.

It appears that the graph ends at $y = 7$ on the middle piece of the graph but this is suppressed by the far left graph since the output of that subfunction grows to ∞ there! Since we have a closed circle at $y = -6$ and an open circle at $y = -2$, the range is $(-\infty, -6] \cup (-2, +\infty)$.

Zeros: It is clear that the graph has only one x-intercept appearing on the graph of the first subfunction $y = -2x-8$. Setting $y = 0$ and solving for x, we easily obtain the zero $x = -4$.

Increasing interval: $(-2, 2)$.
Decreasing intervals: $(-\infty, -3)$ and $(2, +\infty)$.

1.6 Parent Functions



Identity function: $y = x$

Domain :

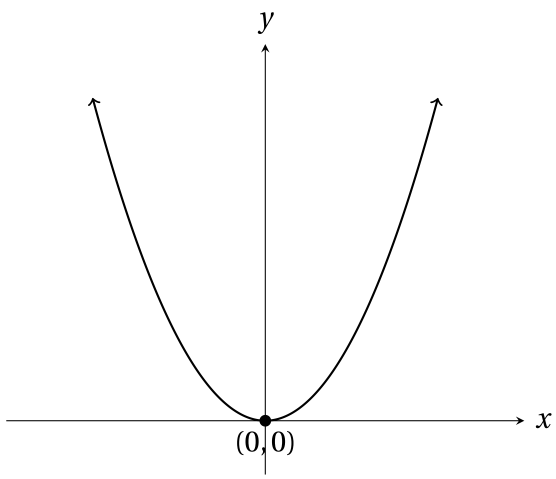
Range :

Zeros :

Increasing interval :

Decreasing interval :

Symmetry :



(Simple) Quadratic function: $y = x^2$

Domain :

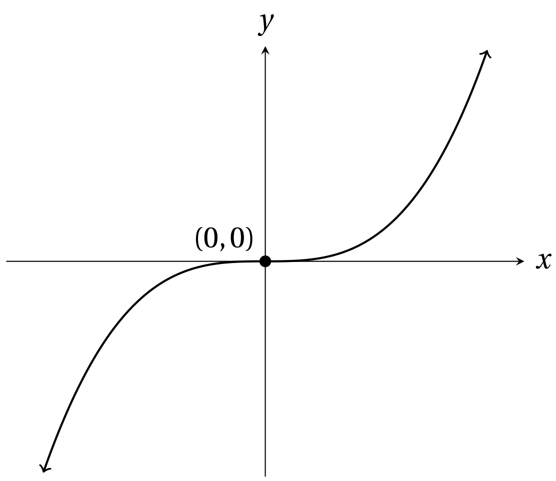
Range :

Zeros :

Increasing interval:

Decreasing interval:

Symmetry:



(Simple) Cubic function: $y = x^3$

Domain :

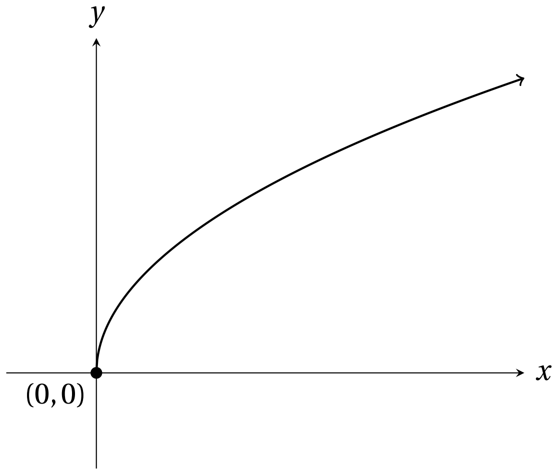
Range :

Zeros :

Increasing interval:

Decreasing interval:

Symmetry:



Square root function: $y = \sqrt{x}$

Domain :

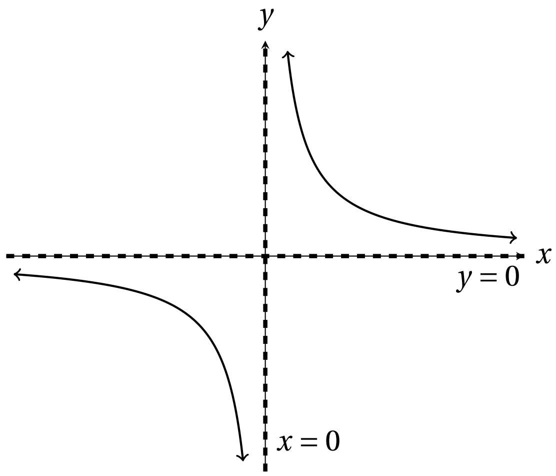
Range :

Zeros :

Increasing interval:

Decreasing interval:

Symmetry:



Reciprocal function: $y = \dfrac{1}{x}$

Domain :

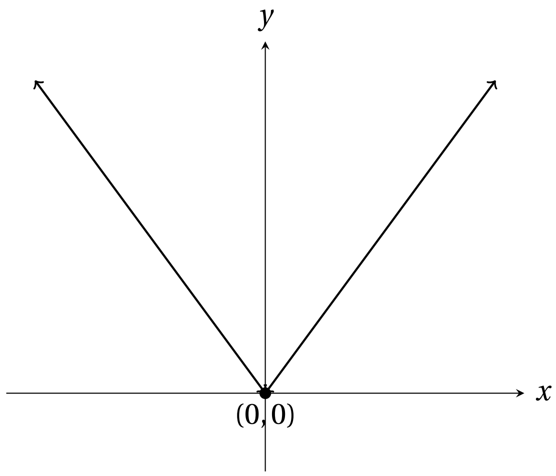
Range :

Zeros :

Increasing interval:

Decreasing interval:

Symmetry:



Absolute value function: $y = |x|$

Domain :

Range :

Zeros :

Increasing interval:

Decreasing interval:

Symmetry:

1.7 Transformation of Functions

Many functions can be treated as transformation of parent functions from Section 1.6. For this reason, graphs of functions can be obtained by suitably transforming graphs of parent functions. To better understanding the concept of transformations, we investigate the square root function $y = \sqrt{x}$ and possible transformations acting on it.

Function	$x = -3$	$x = -2$	$x = -1$	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$f(x) = \sqrt{x}$							
$f(x+2) = \sqrt{x+2}$							
$f(x-2) = \sqrt{x-2}$							
$f(x)+2 = \sqrt{x}+2$							
$f(x)-2 = \sqrt{x}-2$							
$f(-x) = \sqrt{-x}$							
$-f(x) = -\sqrt{x}$							

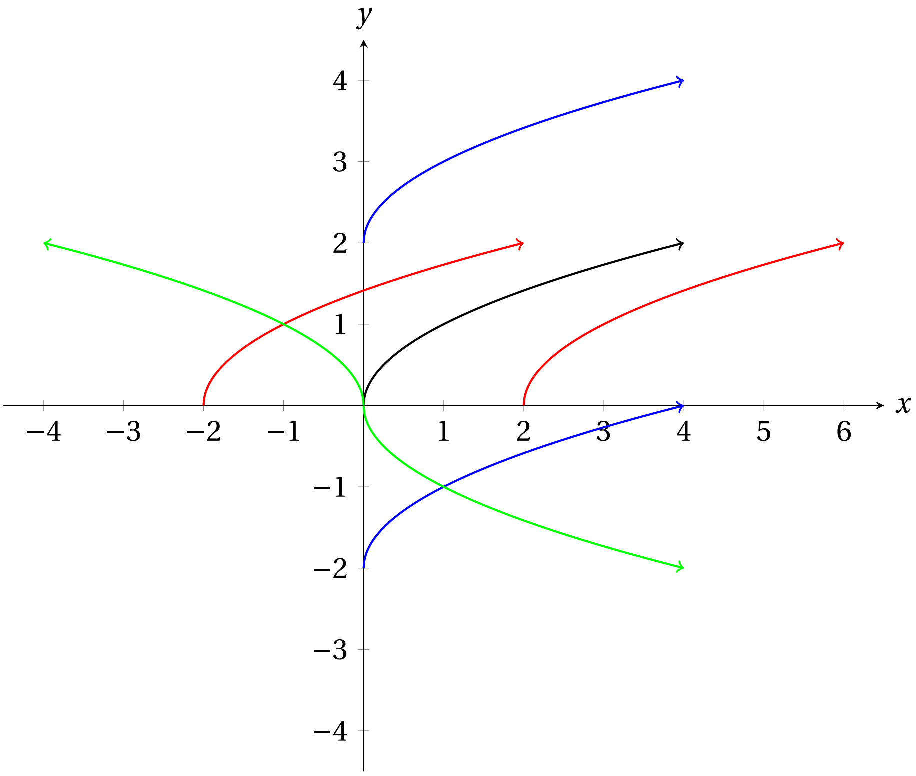


Figure 1: **Rigid transformations** of $y = \sqrt{x}$, corresponding to **translation (shifting)** or **reflection**.

Now, there is one important pattern hidden in the example above.

1. If the transformation involves changing the ____________________, then the graph changes in the ____________________ direction.
2. If the transformation involves changing the ____________________, then the graph changes in the ____________________ direction.

Function	$x=-2$	$x=-1$	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$
$f(x)=\sqrt{x}$							
$f(2x)=\sqrt{2x}$							
$f\left(\frac{x}{2}\right)=\sqrt{\frac{x}{2}}$							
$2f(x)=2\sqrt{x}$							
$\frac{f(x)}{2}=\frac{\sqrt{x}}{2}$							

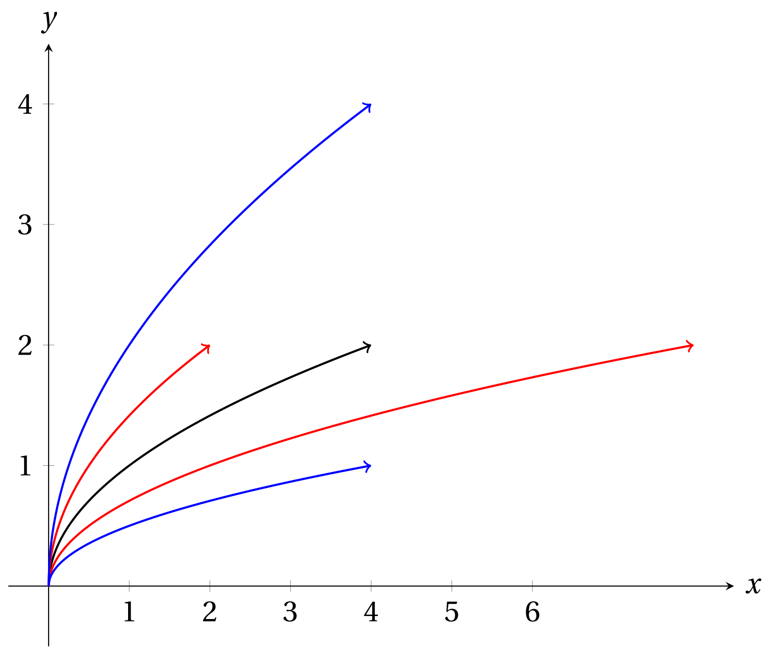


Figure 2: **Nonrigid transformations** of $y=\sqrt{x}$, corresponding to **stretching** or **shrinking**.

We summarise all the transformations in the following table. Here, $c>0$ is a positive number.

Transformations	Direction of transformations	Ranges of c	Action
$y=f(x+c)$	Horizontal		Shift c units to the left
$y=f(x-c)$	Horizontal		Shift c units to the right
$y=f(x)+c$	Vertical		Shift c units upwards
$y=f(x)-c$	Vertical		Shift c units downwards
$y=f(-x)$	Horizontal	- - -	Reflect about y-axis
$y=-f(x)$	Vertical	- - -	Reflect about x-axis
$y=f(cx)$	Horizontal	$0<c<1$	Stretch
	Horizontal	$c>1$	Shrink
$y=cf(x)$	Vertical	$0<c<1$	Shrink
	Vertical	$c>1$	Stretch

Here is one little annoying observation: Note that every transformation in the table above has x instead of $-x$. This means that none of the above can be applied in a straightforward manner if we have an $-x$ term in a given function. One way is to factor out the negative sign in order to apply any transformation rules above. We will demonstrate this in Example 1.26.

Below is a useful **rule of thumb** for sketching graphs of functions $f(x)$ using transformations.

1. Identify the parent function $g(x)$.
2. Describe all transformations from the parent function to the given function. Loosely speaking, the idea is to start with the transformation involving the domain/input first and only then we deal with transformation involving range/output.
3. Write $f(x)$ in terms of $g(x)$ and sketch $f(x)$.

Example 1.23. Sketch the graph of $f(x) = (-x)^3 + 4$.

Example 1.24. Sketch the graph of $f(x) = 2 - (x+5)^2$.

Example 1.25. Sketch the graph of $f(x) = \sqrt{7-x} - 3$.

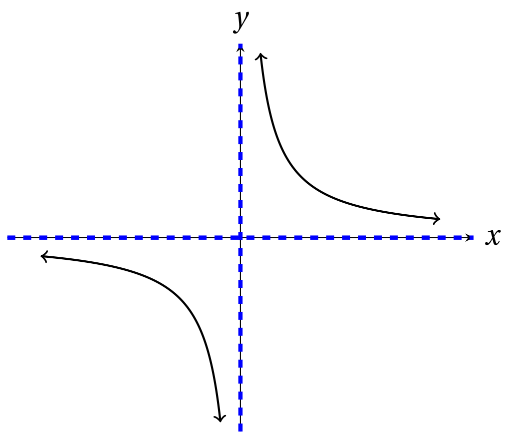
Example 1.26. Sketch the graph of $f(x) = \dfrac{1}{5-x} + 2$.

Solution: Since there is an $-x$ term appearing in $f(x)$, we factor out the negative sign as follows:

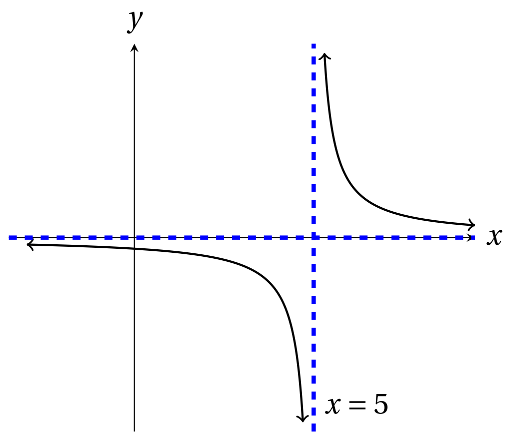
$$f(x) = \frac{1}{5-x} + 2 = \frac{1}{-(x-5)} + 2.$$

The parent function is the reciprocal function $g(x) = \dfrac{1}{x}$. The graph of $f(x)$ is obtained as follows:

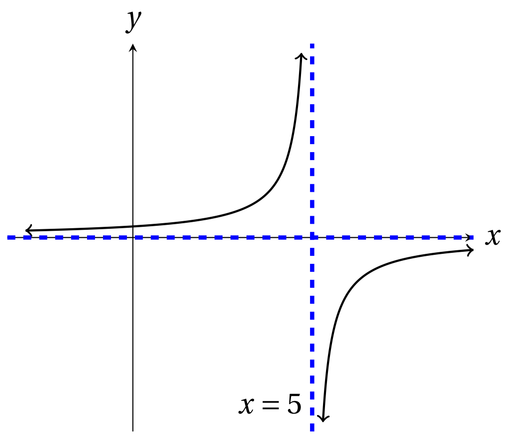
1. Shift the graph of $g(x)$, 5 units to the right;
2. Reflect the graph of $g(x-5)$ about the vertical line $x = 5$; (Note that we do not reflect about the y-axis since the negative sign is being applied to $(x-5)$ but not x.)
3. Shift the graph of $g(-(x-5))$, 2 units upward.



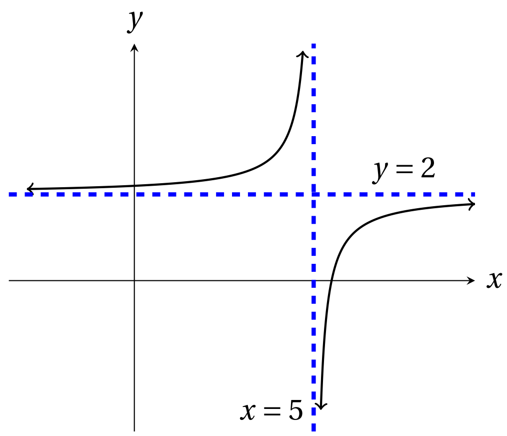
Graph of $g(x)$.



1. Graph of $g(x-5)$.



2. Graph of $g(-(x-5))$.



3. Graph of $g(-(x-5)) + 2 = f(x)$.

As an exercise, check that the x-intercept is $(11/2, 0)$ and the y-intercept is $(0, 11/5)$.

Example 1.27. Sketch the graph of $f(x) = (2x)^2$, $f(x) = 2x^2$ and $f(x) = \left(\frac{x}{3}\right)^3$.

1.8 Combination of Functions: Composite Functions

We begin with *arithmetic combination* of functions f and g, consisting of addition, subtraction, multiplication and division. Note that the domain of these arithmetic combinations consists of all real numbers that belong to both $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$, *i.e.* the intersection of $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$.

Example 1.28. Given $f(x) = x^3 - 2x$ and $g(x) = x - 4$, find $f + g$, $f - g$, fg, f/g and their respective domains.

Another combination of two functions is to form the composition of one with the other. The **composition** of the function f with the function g is given by

$$(f \circ g)(x) = f(g(x)).$$

The notation $f \circ g$ reads "f composed with g". The domain of $f \circ g$ is the set of all x in $\mathrm{dom}(g)$ such that $g(x)$ is in $\mathrm{dom}(f)$.

Example 1.29. Compute $f \circ g$ and $g \circ f$ for the following pairs of functions.

(a) $f(x) = x^2 + 2x$, $g(x) = x - 1$

(b) $f(x) = \dfrac{1}{x}$, $g(x) = x^2 - 3x$

Domain of composition of functions

The rule of thumb for composition of functions is go from right to left! To find the domain of $f \circ g$, we look at the following chain of relation:

$$\operatorname{dom}(g) \longrightarrow \operatorname{range}(g) \longrightarrow \operatorname{dom}(f).$$

- If $\operatorname{dom}(f) = (-\infty, \infty)$, then $\operatorname{dom}(f \circ g) = \operatorname{dom}(g)$.
- If $\operatorname{range}(g)$ is contained completely in $\operatorname{dom}(f)$, then $\operatorname{dom}(f \circ g) = \operatorname{dom}(g)$.
- If $\operatorname{range}(g)$ is not contained completely in $\operatorname{dom}(f)$, then we need to find the "bad values", *i.e.* all x-values such that f is not defined at $g(x)$, and remove this from $\operatorname{dom}(g)$.

For example, $[0,4)$ is contained completely in $[-2,6]$ but $[4,\infty)$ is not contained completely in $[-2,6]$.

Example 1.30. Find the domain of $f \circ g$ for the following pairs of functions.

(a) $f(x) = x^2$ and $g(x) = 4 - x^2$

(b) $f(x) = \sqrt{x+4}$ and $g(x) = x^2$

(c) $f(x) = \dfrac{1}{x}$ and $g(x) = x + 3$

1.9 Inverse Functions

As you probably know from previous math courses, the inverse of a function $y = f(x)$ is found by "swapping x and y" and solve for y. This makes sense because the purpose of the inverse function is to reverse the operations of f, by changing input to output and output to input. For the remaining section, you should try to **associate inverse functions with the word "swapping"** because this keyword will serve as a foundation for many development of inverse functions later. Let us explore the geometrical meaning of this swapping process.

Graphically

Consider the function of $y = f(x) = 2x - 4$. We find its inverse by reversing the operations of f. What the function f does to a given input x is the following:

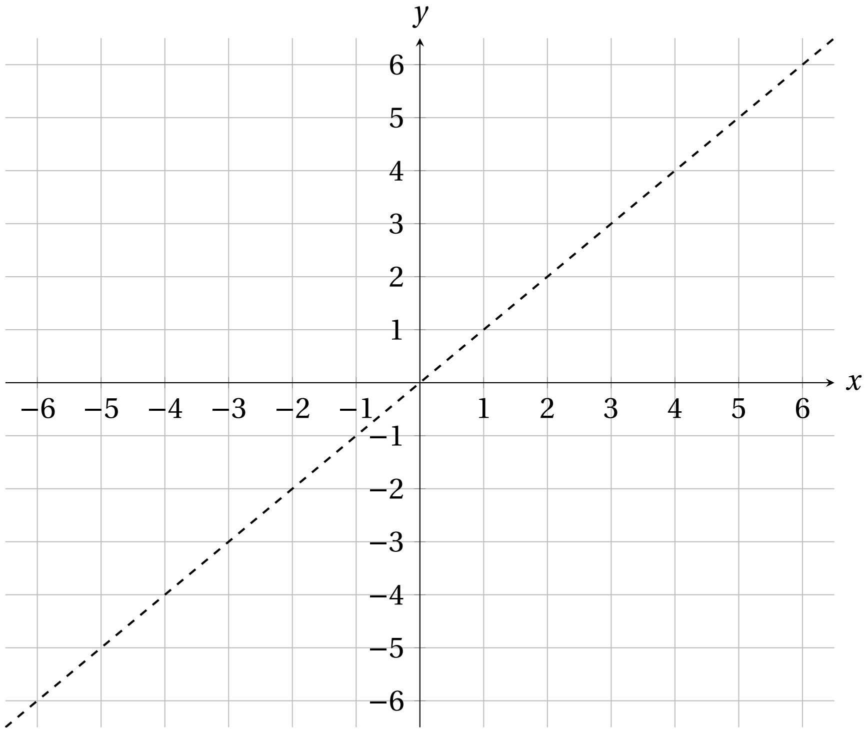
1. Multiply x by 2 to get $2x$;
2. Subtract 4 from $2x$ to get $2x - 4$.

To reverse these operations, we should

1.

2.

This suggests that the inverse of f should be $y =$ ____________________. Let us verify this by finding a few solutions of $y = 2x - 4$, swap coordinates and check if they satisfy the inverse function we found above.



This provides a graphical method of determining if a function has an inverse. Reflect the graph of f about the line $y = x$ to obtain a new graph. If this new graph is a function (passes the Vertical Line Test), then f has an inverse. With the keyword swapping, we obtain the **Horizontal Line Test**:

> A function f has an inverse function
>
> $\Updownarrow$
>
> Any horizontal line intersects the graph of f at most one point.

If you have been paying attention until now, you should be able to answer the following question: Does the function $f(x) = x^2$ has an inverse?

Definition 1.31. Given a function $f(x)$, its **inverse $f^{-1}(x)$**, if it exists, is a function from range(f) to dom(f) such that

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$$
$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x.$$

We must have dom(f) = range(f^{-1}) and range(f) = dom(f^{-1}); this must be true so that the swapping process is valid! A function f has an inverse function if and only if f is **one-to-one**.

Algebraically

As tempting as the reflection method may seem, it is not that practical in terms of computing the inverse function f^{-1} because we must sketch the graph of f first and that itself might be difficult in general. With the swapping again, we can find the inverse of a function f algebraically as follows:

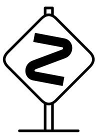
1. Set $y = f(x)$, then swap the variable x with the variable y and solve for y.

2. Once we found y, set $y = f^{-1}(x)$.

Example 1.32. Find the inverse function of $f(x) = \dfrac{3x+4}{5}$ and verify it using definition of inverse function.

Example 1.33. Find the inverse function of $f(x) = \sqrt{x+1} - 5$.

Example 1.34. Find the inverse function of $f(x) = \dfrac{3x+4}{x-5}$.



If there is more than one choice for the inverse, then choose the one such that it satisfies

$$f^{-1}\colon \text{range}(f) \longrightarrow \text{dom}(f),$$

e.g. dom(f) must equal range(f^{-1}).

Example 1.35. Find the inverse function of $f(x) = x^2+2, x \geq 0$.

Solution: For $f(x) = x^2+2 = y$, we have:

$$\begin{aligned} y^2+2 &= x \\ y^2 &= x-2 && \left[\text{Substract 2 from each side.}\right] \\ y &= \pm\sqrt{x-2}. && \left[\text{Take the square root of each side.}\right] \end{aligned}$$

Since dom$(f) = [0,\infty)$ and dom(f) must equal range(f^{-1}), it follows that the inverse of $f(x)$ must be the positive square root and

$$f^{-1}(x) = \sqrt{x-2}.$$

2 Polynomials and Rational Functions

At this stage of the course, you should be comfortable with linear functions and sketching graphs using transformations. Let's move on to the next topic: polynomials and rational functions. The ultimate goal here is to learn how to sketch their graphs, including some of their important features. For those of you who are searching for motivation, polynomials are the base-level object in a wide variety of sciences; you can never avoid them no matter where you go or what you do. Recall the parent functions $y = x^2$ and $y = x^3$. In general, we can replace the power with any nonnegative integer $\{0,1,2,\ldots\}$ and to make things more exciting, we may add, subtract or multiply these power functions which results in polynomials.

Definition 2.1. A **polynomial function with degree *n*** has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \quad \text{such that } a_n \neq 0.$$

There are two important numbers associated with polynomials that somewhat reveal the shape of the graph of polynomials.

1. The largest (highest) power among all individual x-terms, called the **degree**.
2. The coefficient in front of the largest power of x, called the **leading coefficient**.

Examples are an integral part in this source and they are as important as the definitions. For this reason, I have crammed these notes with examples. When you review for your exams, don't just flick through them saying, "yeah, yeah, that's obvious". Make a serious effort to study them and know them for the exam.

2.1 Quadratic Functions

I gathered that you have seen quadratic functions in previous math courses so this section should be a review for many of you. A **quadratic function** is a polynomial of degree 2, with general form

$$f(x) = ax^2 + bx + c.$$

The graph of a quadratic function is a U-shaped curve called a **parabola** and its shape depends on the leading coefficient $a \neq 0$:

To sketch the graph of quadratic equations, we rewrite $f(x) = ax^2 + bx + c$ in terms of the **standard form**

$$\boxed{f(x) = a(x-h)^2 + k,}$$

where

1. the vertical line $x = h$ is the **axis of symmetry** or simply the **axis** of the parabola; (Can you tell why we called it the axis of symmetry?)
2. the point (h, k) is the **vertex** of the parabola, which is the lowest point (minimum) if $a > 0$ and the highest point (maximum) if $a < 0$.

Watch out for the $-h$ term in the standard form $f(x) = a(x-h)^2 + k$. This is a common mistake among students!

Example 2.2. Write the standard form of the equation of the parabola whose vertex is $(1, -2)$ and that passes through the point $(-1, 14)$. Find the x- and y-intercepts and sketch the parabola.

Remark 2.3. The standard way to achieve the standard form is by **completing the square**. But there is a simple formula that allows us to immediately write down the vertex:

$$\text{vertex} = (h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$$

I can never remember this but you might...........

Example 2.4. Find the vertex and axis of symmetry for the following quadratic functions.

(a) $f(x) = x^2 + 8x + 7$

(b) $f(x) = 3x^2 - 6x - 5$

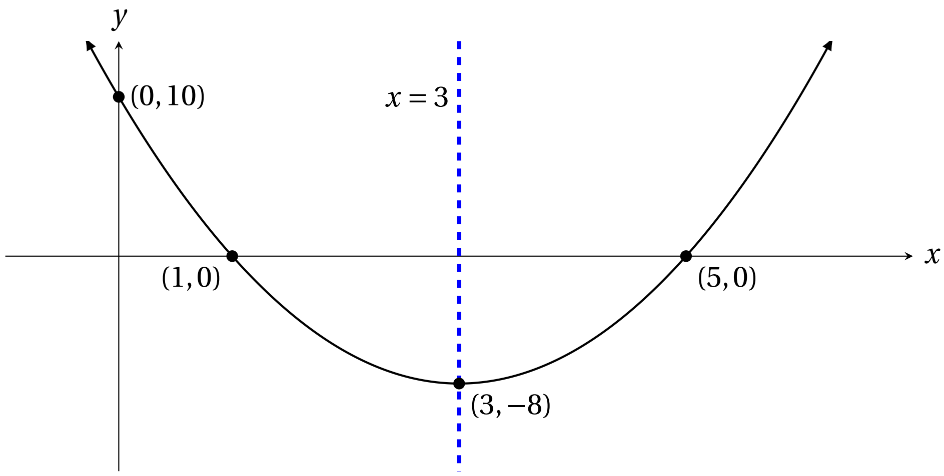
(c) $f(x) = 4x^2 + 24x - 3$

Example 2.5. Given the function $f(x) = 2x^2 - 12x + 10$, find the vertex, axis of symmetry and x- and y-intercepts. Sketch its graph and clearly label its vertex, axis of symmetry and the intercepts.

Solution: We factor out the leading coefficient $a = 2$ from the first two terms before complete the square:

$$\begin{aligned} f(x) = 2x^2 - 12x - 3 &= 2\left[x^2 - 6x\right] + 10 \\ &= 2\left[x^2 - 6x + 3^2 - 3^2\right] + 10 \\ &= 2\left[(x-3)^2 - 9\right] + 10 \\ &= 2(x-3)^2 - 18 + 10 \\ &= 2(x-3)^2 - 8. \end{aligned}$$

Thus, the vertex of the graph of $f(x)$ is $(3, -8)$ and the axis of symmetry is $x = 3$. Setting $x = 0$ in $f(x)$ gives the y-intercept $(0, 10)$. You should check that the x-intercepts are $(1, 0)$ and $(5, 0)$.



2.2 Polynomials of Higher Degree

To graph polynomials, the first step is to determine the **end (limiting) behaviour** of the given polynomials. By end behaviour we mean, what is the limiting values of $f(x)$ as we increases x to $+\infty$ or decreases x to $-\infty$? To this end, it is useful to introduce the **limit notation**:

$f(x) \longrightarrow +\infty$ as $x \longrightarrow +\infty$ reads "$f(x)$ goes to $+\infty$ as we increases x to $+\infty$".
$f(x) \longrightarrow -\infty$ as $x \longrightarrow +\infty$ reads "$f(x)$ goes to $-\infty$ as we increases x to $+\infty$".
$f(x) \longrightarrow +\infty$ as $x \longrightarrow -\infty$ reads "$f(x)$ goes to $+\infty$ as we decreases x to $-\infty$".
$f(x) \longrightarrow -\infty$ as $x \longrightarrow +\infty$ reads "$f(x)$ goes to $-\infty$ as we decreases x to $-\infty$".

Let us explore this problem by studying these simple polynomials: $\pm x, \pm x^2, \pm x^3, \pm x^4$.

The analysis reveals that the end behaviour depends on both the degree n and the sign of leading coefficient a_n. This leads to the **Leading Coefficient Test**:

Remark 2.6. Always refer back to the case of linear and quadratic functions if you forget about this test, or better , my awesome hand-dance.

Example 2.7. Find the end behaviour of the following polynomials.

(a) $f(x) = 200x^5 - 3x^4 + x^2 - 23$

- As $x \longrightarrow \infty$, $f(x) \longrightarrow$ ________________.
- As $x \longrightarrow -\infty$, $f(x) \longrightarrow$ ________________.

(b) $f(x) = x^3 - 5x^4 - 36x^2 + 4x + 4$

- As $x \longrightarrow \infty$, $f(x) \longrightarrow$ ________________.
- As $x \longrightarrow -\infty$, $f(x) \longrightarrow$ ________________.

(c) $f(x) = -x(x-5)^3(x+2)^2(x-1)^5$

- As $x \longrightarrow \infty$, $f(x) \longrightarrow$ ________________.
- As $x \longrightarrow -\infty$, $f(x) \longrightarrow$ ________________.

(d) $f(x) = x^2(x+4)(x-1)^4(x-7)^3$

- As $x \longrightarrow \infty$, $f(x) \longrightarrow$ ________________.
- As $x \longrightarrow -\infty$, $f(x) \longrightarrow$ ________________.

Real zeros/roots of polynomial functions

Because polynomials are also functions, finding zeros of polynomials is equivalent to finding zeros of functions. Zeros are also called roots, and these two terms will be used interchangeably throughout the course. For completeness, we restate the definition of zeros of functions and include other equivalent statements.

Given a polynomial function f and a real number a, the following statements are equivalent:

1. $x = a$ is a **real zero** or **real root** of the function $f(x)$.
2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a **linear factor** of the polynomial $f(x)$.
4. $(a, 0)$ is an x-*intercept* of the graph of $f(x)$.

The last statement is crucial for us, because the second step in graphing polynomials is to **locate the x-intercepts** and from above this amounts to **finding all the real roots**.

Example 2.8. Find all real roots of the given polynomials.

(a) $f(x) = x^4 - x^3 - 20x^2$

(b) $f(x) = 4x^4 - 9x^2$

2.3 Polynomial and Synthetic Division

So far we have only learned how to solve quadratic equations, either by factoring or applying quadratic formula, but what about polynomials of degree $3,4$ or higher? The approach is rather simple: we try to factor polynomials until they reduce to a quadratic equation, which is a familiar territory, so the question is **how do we factor polynomials?** As you could have guess from the title, we perform polynomial division.

Synthetic division

Long division is the cumbersome one so I entrust the review-task to yourself. Now, synthetic division is the fast-track-polynomial-division, that unfortunately, only works for linear divisors of the form $(x-a)$; nonetheless it suffices for the course. Attempting to describe synthetic division in words is honestly impractical so let us jump straight into some examples!

Example 2.9. Use synthetic division to divide $f(x) = 2x^3 - 8x^2 + 3x - 9$ by $x+1$.

Example 2.10. Use synthetic division to divide $f(x) = 3x^3 + 5x^2 - 3x + 27$ by $x+3$.

In general, given a polynomial $f(x)$ and a linear divisor $(x-a)$, we have that

$$f(x) = (x-a)g(x) + \text{remainder},$$

where the quotient $g(x)$ is obtained using synthetic division. If $(x-a)$ is a factor of $f(x)$, then the remainder must be zero. Otherwise, the remainder is nonzero and equal to $f(a)$. This provide another means of finding $f(a)$: Instead of directly evaluating $f(a)$, we may compute $f(a)$ using synthetic division on $(x-a)$.

Example 2.11. Find $f(3)$ for the following polynomials.

(a) $f(x) = x^3 - 27$

(b) $f(x) = 2x^4 - 5x^3 - 4$

Example 2.12. Use synthetic division to divide $f(x) = 4x^4 + 5x^3 - 5x^2 + x + 3$ by $x+2$.

Solution:

$$\begin{array}{r|rrrrr} & 4 & 5 & -5 & 1 & 3 \\ -2 & & -8 & 6 & -2 & 2 \\ \hline & 4 & -3 & 1 & -1 & 5 \end{array}$$

Since the remainder is 5,

$$f(x) = (x+2)(4x^3 - 3x^2 + x - 1) + 5,$$

or

$$\frac{f(x)}{x+2} = \frac{4x^4 + 5x^3 - 5x^2 + x + 3}{x+2} = (4x^3 - 3x^2 + x - 1) + \frac{5}{x+2}.$$

Repeated roots

After we completely factor the polynomials to obtain the zeros (and hence the x-intercepts), the final task is to determine whether the graph "crosses" or "touches" these x-intercepts. Now I am just throwing big words to you aren't I? Let me illustrate what these mean with two simple examples.

Example 2.13. Sketch the graph of $f(x) = x^3 - x^2 - 30x$. Clearly label any intercepts.

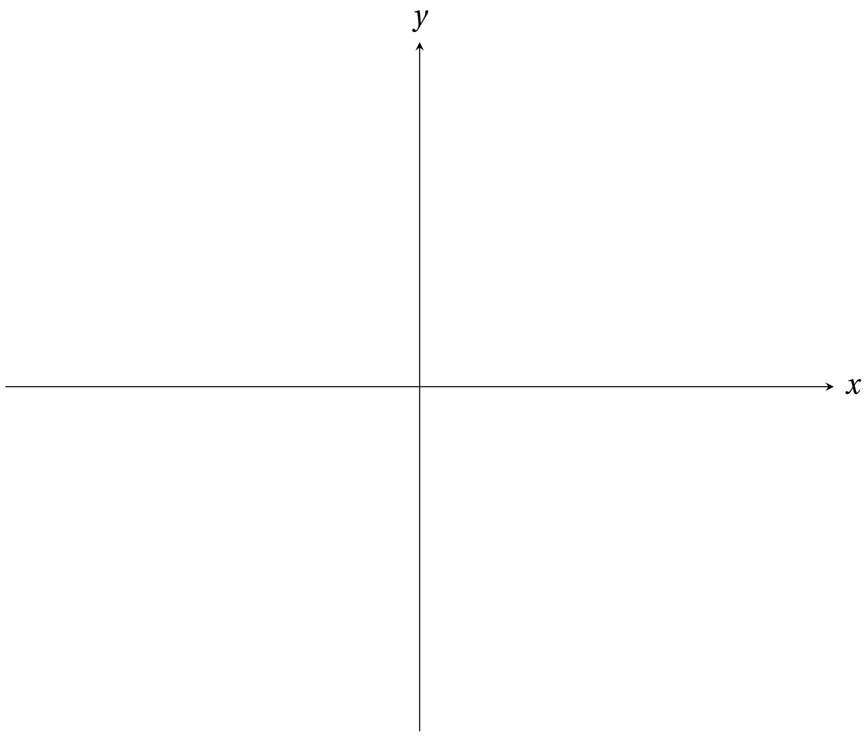
Example 2.14. Sketch the graph of $f(x) = x^4 - x^3 - 30x^2$. Clearly label any intercepts .

As you can see, whether the graph crosses or touches the x-intercept $(a,0)$ depends on the **multiplicity** of the root $x=a$, *i.e.* how many times it repeats.

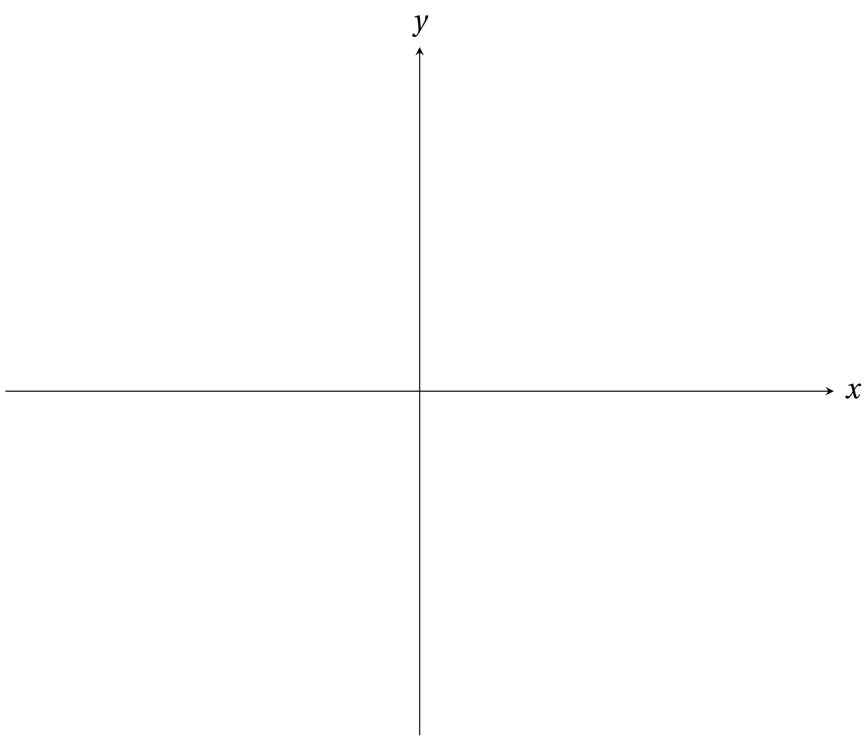
1. If the root $x=a$ repeats **odd** number of times, then the graph **crosses** the x-axis at $x=a$.

2. If the root $x=a$ repeats **even** number of times, then the graph **touches** the x-axis at $x=a$.

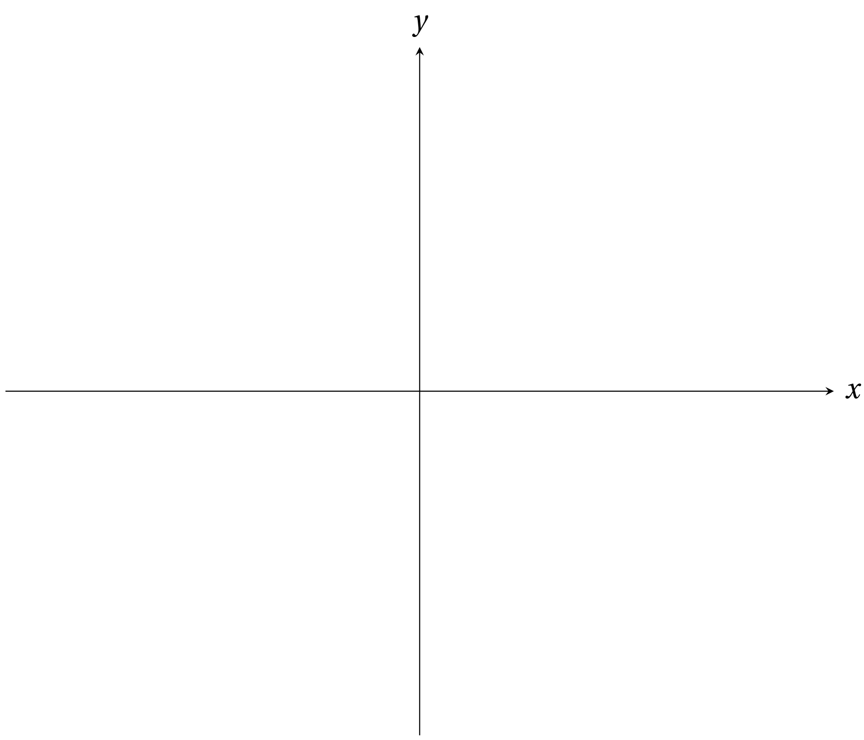
Example 2.15. Suppose the polynomial $f(x)=2x^3+x^2-5x+2$ has a linear factor $(x+2)$. Find the real roots of $f(x)$ and sketch the graph of $f(x)$. Clearly label any intercepts and indicate with arrows the end behaviour of $f(x)$. Where is $f(x)>0$?



Example 2.16. Suppose the polynomial $f(x) = x^3 - x^2 - 5x - 3$ has a zero at $x = -1$. Find the linear factors of $f(x)$ and sketch the graph of $f(x)$. Clearly label any intercepts and indicate with arrows the end behaviour of $f(x)$. Where is $f(x) < 0$?



Example 2.17. Use synthetic division to show that $x = 1$ and $x = -4$ are solutions to the equation $f(x) = x^4 - x^3 - 13x^2 + 25x - 12 = 0$. Find all real roots of $f(x)$ and sketch the graph of $f(x)$. Clearly label any intercepts and indicate with arrows the end behaviour of $f(x)$. Where is $f(x) \geq 0$?



Example 2.18. Suppose $(x+3)$ and $(x-1)$ are two linear factors of the equation $f(x) = x^4+3x^3-7x^2-15x+18 = 0$. Find all real roots of $f(x)$ and sketch the graph of $f(x)$. Clearly label any intercepts and indicate with arrows the end behaviour of $f(x)$. Where is $f(x) \leq 0$?

Solution: It was given in the question that $x = -3$ and $x = 1$ are two real roots of $f(x)$. We use synthetic division to find the remaining real roots:

$$\begin{array}{r|rrrrr} & 1 & 3 & -7 & -15 & 18 \\ -3 & & -3 & 0 & 21 & -18 \\ \hline & 1 & 0 & -7 & 6 & 0 \end{array}$$

$$\begin{array}{r|rrrr} & 1 & 0 & -7 & 6 \\ 1 & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

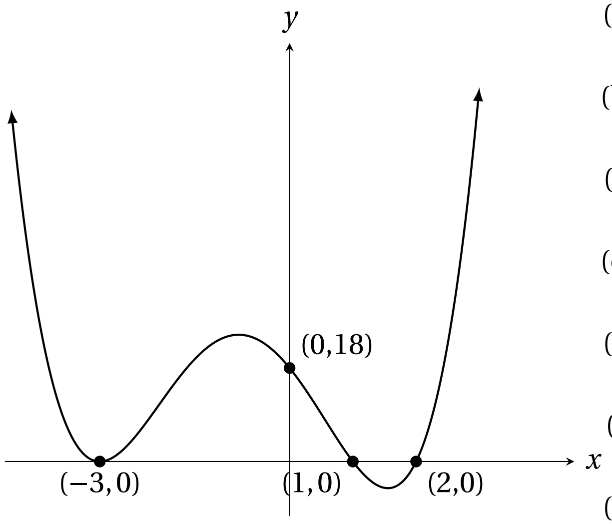
Thus, $f(x)$ can be written as as

$$f(x) = (x+3)(x-1)(x^2+x-6) = (x+3)(x-1)(x+3)(x-2),$$

and so the real roots of $f(x)$ are $x = -3, 1, 2$. The x-intercepts are $(-3,0), (1,0), (2,0)$. Setting $x = 0$ gives the y-intercept $(0,18)$. Since the degree of $f(x)$ is 4 which is even and the leading coefficient of $f(x)$ is 1 which is positive, the end behaviour of $f(x)$ is as follows:

$$f(x) \longrightarrow +\infty \quad \text{as } x \longrightarrow +\infty$$
$$f(x) \longrightarrow +\infty \quad \text{as } x \longrightarrow -\infty.$$

Finally, the graph of $f(x)$ touches the x-axis at $x = -3$ since the real root $x = -3$ repeats twice.



(a) Real roots: $x = -3, 1, 2$.

(b) Linear factors: $(x+3), (x-1), (x-2)$.

(c) x-intercepts: $(-3,0), (1,0), (2,0)$.

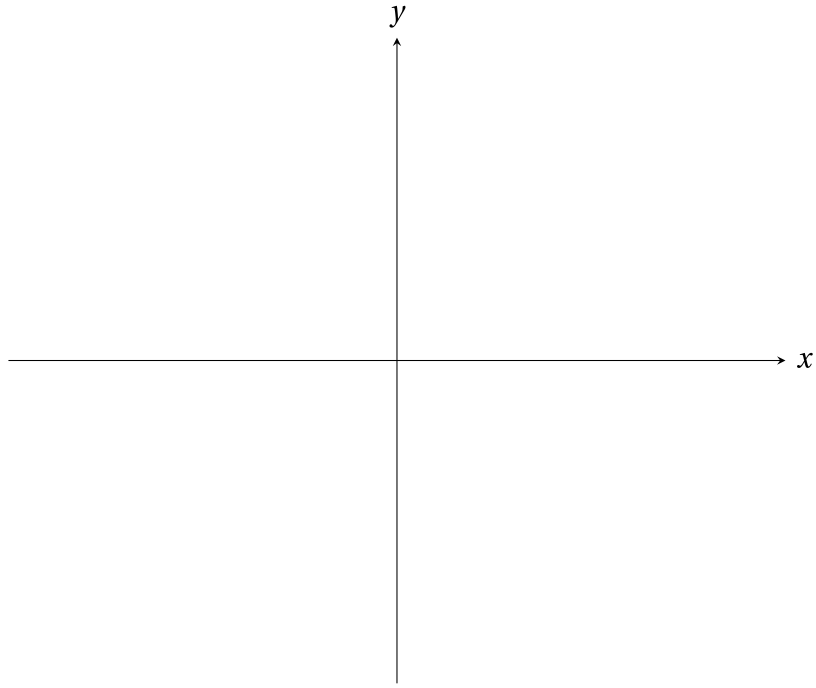
(d) y-intercept: $(0,18)$.

(e) As $x \longrightarrow +\infty$, $f(x) \longrightarrow +\infty$.

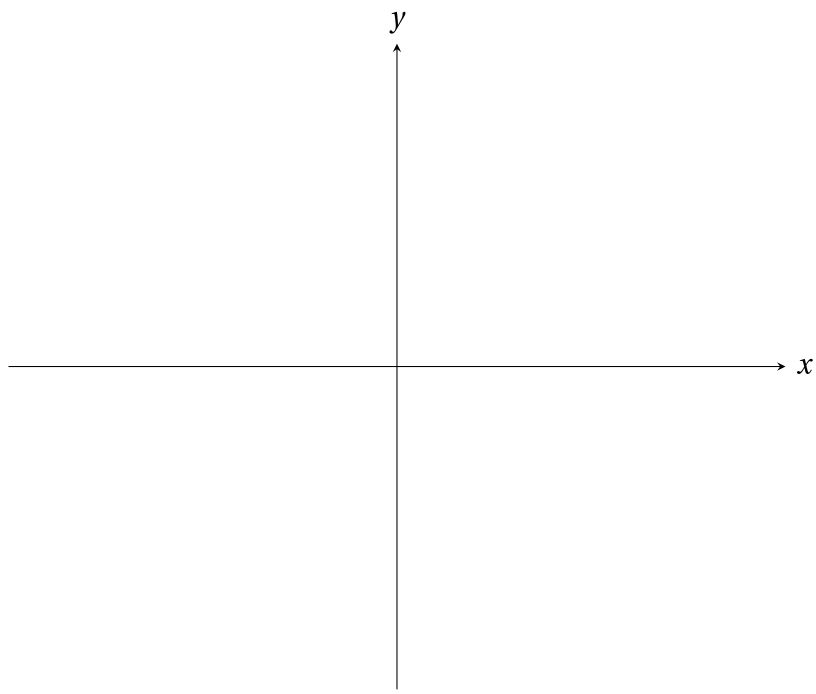
(f) As $x \longrightarrow -\infty$, $f(x) \longrightarrow +\infty$.

(g) $f(x) \leq 0$ at $\{-3\} \cup [1,2]$.

Example 2.19. Find all real zeros of the polynomial $f(x) = (x-3)^3(x+2)^2(x-5)(x+4)^3$ and sketch the graph of $f(x)$. Clearly label any intercepts and indicate with arrows the end behaviour of $f(x)$.



Example 2.20. Find all real zeros of the polynomial $f(x) = -(x-1)^5(x-5)(x+3)^2(x-7)^2$ and sketch the graph of $f(x)$. Clearly label any intercepts and indicate with arrows the end behaviour of $f(x)$.



2.4 Complex Numbers

You might have noticed that I kept emphasising about real roots previously. The counterpart of real roots are **complex roots**, and the goal of this section is to learn how to manipulate **complex numbers** and understand a thing or two about complex roots.

You probably have learned that the equation $x^2 = -1$ has no real solutions, because taking the square root of negative number just doesn't feel right. Well, mathematicians decided to make this a definition by introducing the **imaginary unit**

$$i = \sqrt{-1}, \quad \text{where } i^2 = -1.$$

With this definition, we then have that

$$x^2 + 1 = 0 \implies x^2 = -1 \implies x = \pm\sqrt{-1} = \pm i.$$

I can't quite explain why complex numbers are important, but nevertheless they are important and so we will do them. You will however encounter complex numbers a lot more if you plan to take Math 1060 in the future.

Definition 2.21. The number $a + bi$ is called a **complex number**, where a and b are real numbers.

1. This is called the **standard form** of complex number.
2. a is the **real part** and b is the **imaginary part** of the complex number.
3. If $b = 0$, then $a + 0i = a$ is a real number.
4. If $a = 0$, then $0 + bi = bi$ is called a **pure imaginary number**.

Operations with complex numbers

To add or subtract any two complex numbers, we simply add or subtract the real parts and imaginary parts respectively, *i.e.* we perform addition or subtraction on like terms. For example,

$$(4 + 2i) + (-2 + i) =$$

$$(4 + 2i) - (-2 + i) =$$

To multiply two complex numbers, we simply expand the parentheses and use the definition that $i^2 = -1$. For example,

$$(4 + 2i)(-2 + i) =$$

Given a complex number $a + bi$, its **complex conjugate** is $a - bi$, where we negate its imaginary part. As an example, the complex conjugate of $4 + 2i$ is $4 - 2i$.

Complex conjugate of $5 - 2i$ is

Complex conjugate of $-3 + 7i$ is

Remark 2.22. Note that the product of complex conjugates is a real number:

$$(a+bi)(a-bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2.$$

Finally, to divide two complex numbers, we multiply both the numerator and denominator by the complex conjugate of the denominator. For example,

$$\frac{4+2i}{1-3i} = \frac{(4+2i)}{(1-3i)}\frac{(\qquad\qquad)}{(\qquad\qquad)} =$$

$$=$$

$$\frac{1+i}{5-2i} = \frac{(1+i)}{(5-2i)}\frac{(\qquad\qquad)}{(\qquad\qquad)} =$$

$$=$$

Complex solutions

Example 2.23. Solve $x^2 + 4 = 0$ and $3x^2 + 7 = 0$.

Example 2.24. Solve $x^2 - 2x + 2 = 0$ using the quadratic formula.

2.5 Zeros of Polynomial Functions

The recurring theme so far has been about finding roots of polynomials and by now you should learn to appreciate the significance of zeros in terms of sketching graphs. In Section 2.3, at least one or two zeros is given in all examples and the remaining zeros are found using synthetic division and solving quadratic equations. Actually, we can methodically guess the zeros of polynomials, specifically **rational zeros**. Consider the polynomial function $f(x) = (2x-5)(3x+2)(x-1)$. The zeros are clearly $x = \frac{5}{2}, -\frac{2}{3}, 1$. On the other hand, expanding $f(x)$ we obtain

$$\begin{aligned} f(x) &= (2x-5)(3x+2)(x-1) \\ &= (2x-5)(3x^2-3x+2x-2) \\ &= (2x-5)(3x^2-x-2) \\ &= 6x^3-2x^2-4x-15x^2+5x+10 \\ &= 6x^3-17x^2+x+10. \end{aligned}$$

The purpose of this computation is to reveal the following key observations.

1. Factors of the constant terms [10] appear in the numerator of the roots $\{5,2,1\}$.

2. Factors of the leading coefficients [6] appear in the denominator of the roots $\{2,3,1\}$.

Rational Root Test: Given a polynomial function $f(x)$, let

$$\begin{aligned} p &= \text{factors of constant term} \\ q &= \text{factors of leading coefficient.} \end{aligned}$$

The set of all possible rational roots is all possible combinations of $\frac{p}{q}$.

Example 2.25. Find the rational zeros of $f(x) = x^3-5x^2+2x+8$.

Rational Root Test only provides a list of POSSIBLE rational roots. This means that among all possible combinations of $\{p/q\}$, it might be the case that none of them are actually zeros of polynomials. One example is $f(x) = x^3 - x^2 + 3$.

Example 2.26. Find all the real solutions of the following equations.

(a) $2x^4 + 3x^3 - 16x^2 + 15x - 4 = 0$

(b) $x^4 + 3x^2 - 10 = 0$

Linear Factorisation Theorem: A polynomial $f(x)$ of degree n can be factored precisely into n linear factors

$$f(x) = a_n(x - r_1)(x - r_2)\dots(x - r_n),$$

where $r_1, r_2, \dots, r_n$ are zeros of $f(x)$, possibly complex numbers.

Example 2.27. For the following functions, find all the zeros and write $f(x)$ as the product of linear factors.

(a) $f(x) = x^3 - 3x^2 - 4x + 12$

(b) $f(x) = x^3 - 3x^2 + x + 5$

Example 2.28. In the previous example, the complex zeros are conjugates of each other. This is in fact true only when the polynomial function has real coefficients. In other words, **complex roots always appear in conjugate pairs**.

(a) Find a third degree polynomial function with real coefficients that has 2 and $1-5i$ as zeros.

(b) Find a fourth degree polynomial function with real coefficients that has $0, -3$ and $2i$ as zeros.

Example 2.29. Find all roots of the function $f(x) = x^4+6x^3+9x^2-6x-10$, given that $x=1$ and $x=-1$ are real roots of $f(x)$. Sketch its graph.

We use synthetic division to find the remaining roots.

$$\begin{array}{r|rrrrr} & 1 & 6 & 9 & -6 & -10 \\ 1 & & 1 & 7 & 16 & 10 \\ \hline & 1 & 7 & 16 & 10 & 0 \end{array} \qquad \begin{array}{r|rrrr} & 1 & 7 & 16 & 10 \\ -1 & & -1 & -6 & -10 \\ \hline & 1 & 6 & 10 & 0 \end{array}$$

Thus, $f(x)$ can be written as as

$$f(x) = (x+1)(x-1)(x^2+6x+10).$$

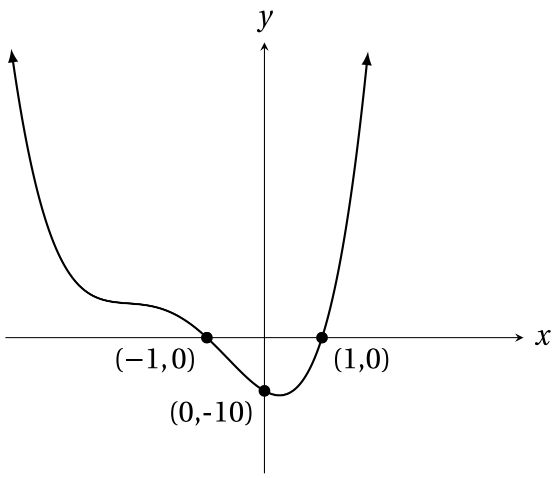
Unfortunately, $x^2+6x+10$ cannot be factored immediately so we need to use the quadratic formula:

$$x = \frac{-6 \pm \sqrt{(-6)^2-4(1)(10)}}{2(1)} = \frac{-6 \pm \sqrt{-4}}{2} = \frac{-6 \pm 2i}{2} = -3 \pm i.$$

Thus the zeros of $f(x)$, including complex ones, are $x = 1, -1, -3+i, -3-i$. The x-intercepts are $(1,0), (-1,0)$. Setting $x=0$ gives the y-intercept $(0,-10)$. Since the degree of $f(x)$ is 4 which is even and the leading coefficient of $f(x)$ is 1 which is positive, the end behaviour of $f(x)$ is as follows:

$$f(x) \longrightarrow +\infty \quad \text{as } x \longrightarrow +\infty$$
$$f(x) \longrightarrow +\infty \quad \text{as } x \longrightarrow -\infty.$$

Finally, the graph of $f(x)$ crosses the x-axis at $x=1$ and $x=-1$.



Complex roots do not correspond to x-intercepts, since they are not real.

2.6 Rational Functions

This section is all about graphing **rational functions** which are quotients of polynomial functions, *i.e.*

$$\text{Rational function} = \frac{\text{polynomial}}{\text{polynomial}} = \frac{N(x)}{D(x)}.$$

Few years ago I wrote a book titled "Principles of Rational Graphs" and my selling point is that if you adhere to these set of instructions below, then you will be able to graph rational functions just like me. **Have I given you hope that you will have a complete mastery of graphing rational functions by the end of the section?**

1. Always **factor** both $N(x)$ and $D(x)$ first. The purpose of this is twofold.
   (a) It is easier to find the **domain** which will be all real numbers except the x-values that result in division by zero.
   (b) Locate **holes**. These are points arising from cancelling common factors of $N(x)$ and $D(x)$.
2. Cancel common factors of $N(x)$ and $D(x)$ before you proceed.
3. Locate x- and y-intercepts.
4. Find all the asymptotes and draw them as dashed lines.
   (a) **Vertical asymptotes (VA)**. These are vertical lines $x = a$ where $f(x) \longrightarrow \pm\infty$ as $x \longrightarrow a$, either from the left or from the right.
   - They correspond to zeros of $D(x)$. (after cancelling common factors!)
   - The graph cannot cross the vertical asymptote.
   - We might have multiple VAs.

   (b) **End behaviour**. This can be found by checking the ratio of largest degree terms of $N(x)$ and $D(x)$. We will either get a **horizontal asymptote (HA)** or a **slant (oblique) asymptote** but not both.

$$\begin{cases} \text{HA } y = 0 & \text{if } \deg(N(x)) < \deg(D(x)), \\ \text{HA } y = \dfrac{\text{leading coefficient of } N(x)}{\text{leading coefficient of } D(x)} & \text{if } \deg(N(x)) = \deg(D(x)), \\ \text{Slant asymptote} & \text{if } \deg(N(x)) > \deg(D(x)). \end{cases}$$

   - Horizontal asymptote is the horizontal line $y = C$ where $f(x) \longrightarrow C$ as $x \longrightarrow \pm\infty$.
   - Slant asymptote is found using polynomial division.
   - It is possible for the graph to cross the horizontal asymptote.
5. Take a sip from your favourite beverage and start graphing. There is one more step..........

The next 1228 examples will be based on these principles, but none of them are useful unless you attend the lectures and listen to me explaining them. If you ever find yourself stressing your friends because you don't know how to graph rational functions, please come and ask me; I promise not to ....

Example 2.30. Sketch the graph of $f(x) = \dfrac{2x-3}{x^2-4}$.

Domain:

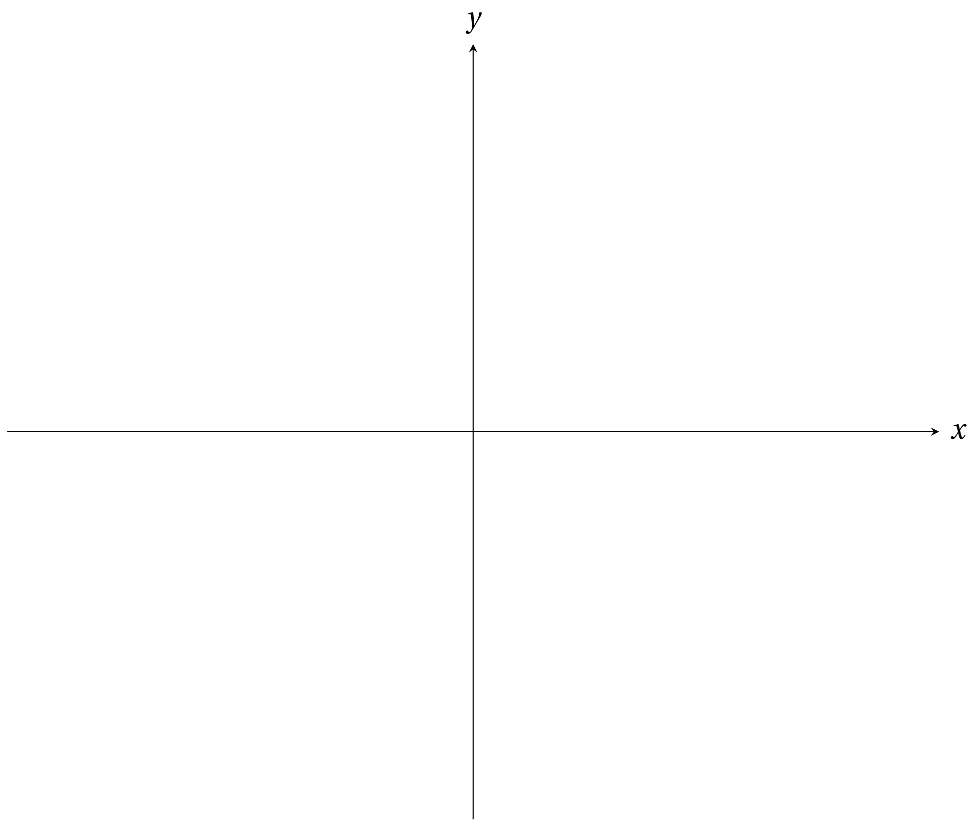
Hole(s):

x-intercept(s):

y-intercept:

VA:

End behaviour:



Example 2.31. Sketch the graph of $f(x) = \dfrac{x-3}{x^2-x-6}$.

Domain:

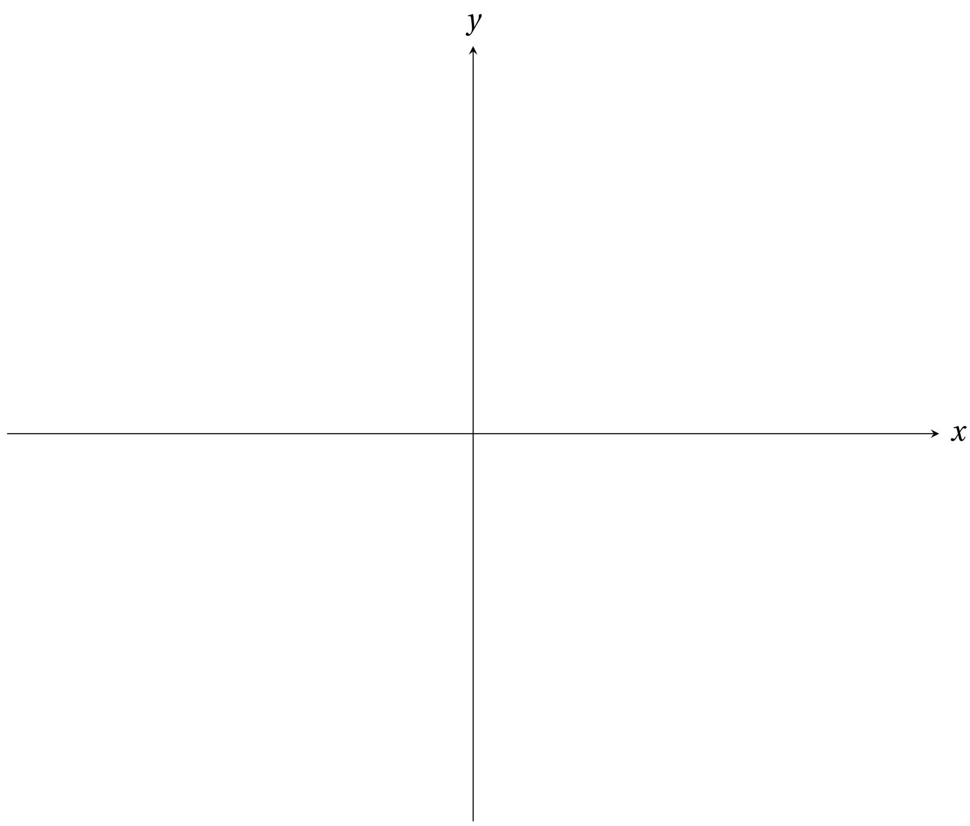
Hole(s):

x-intercept(s):

y-intercept:

VA:

End behaviour:



Example 2.32. Sketch the graph of $f(x) = \dfrac{x+2}{x^2-2x-8}$.

Domain:

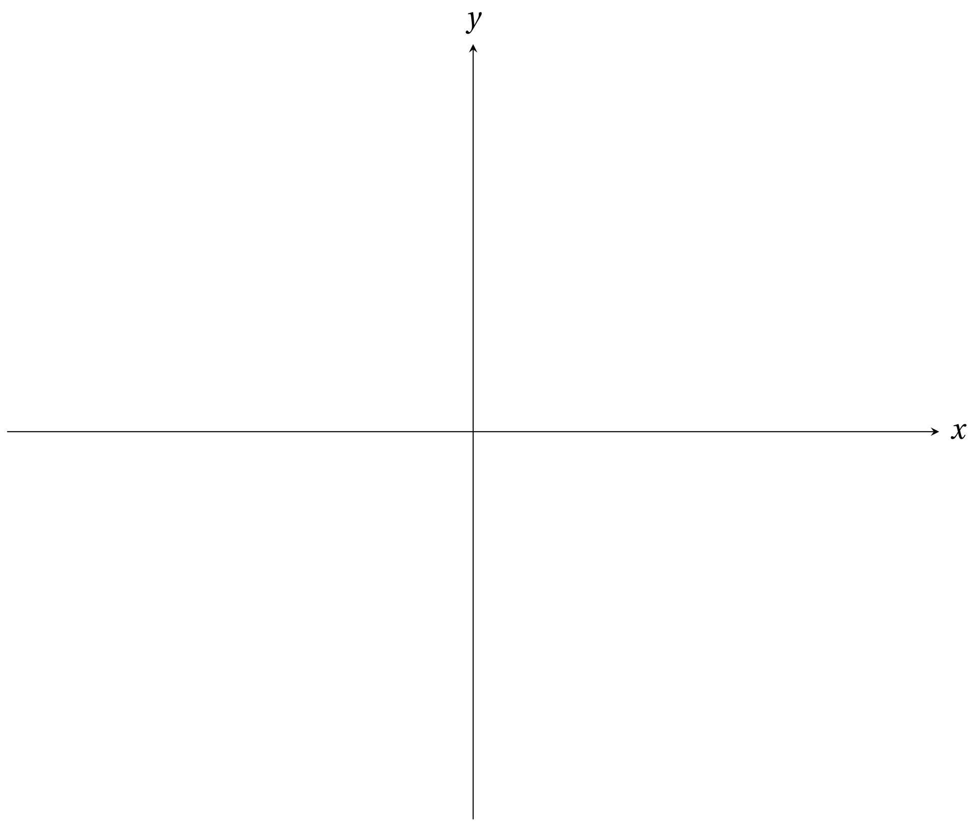
Hole(s):

x-intercept(s):

y-intercept:

VA:

End behaviour:



Example 2.33. Sketch the graph of $f(x) = \dfrac{4x-3}{x^2+2}$.

Domain:

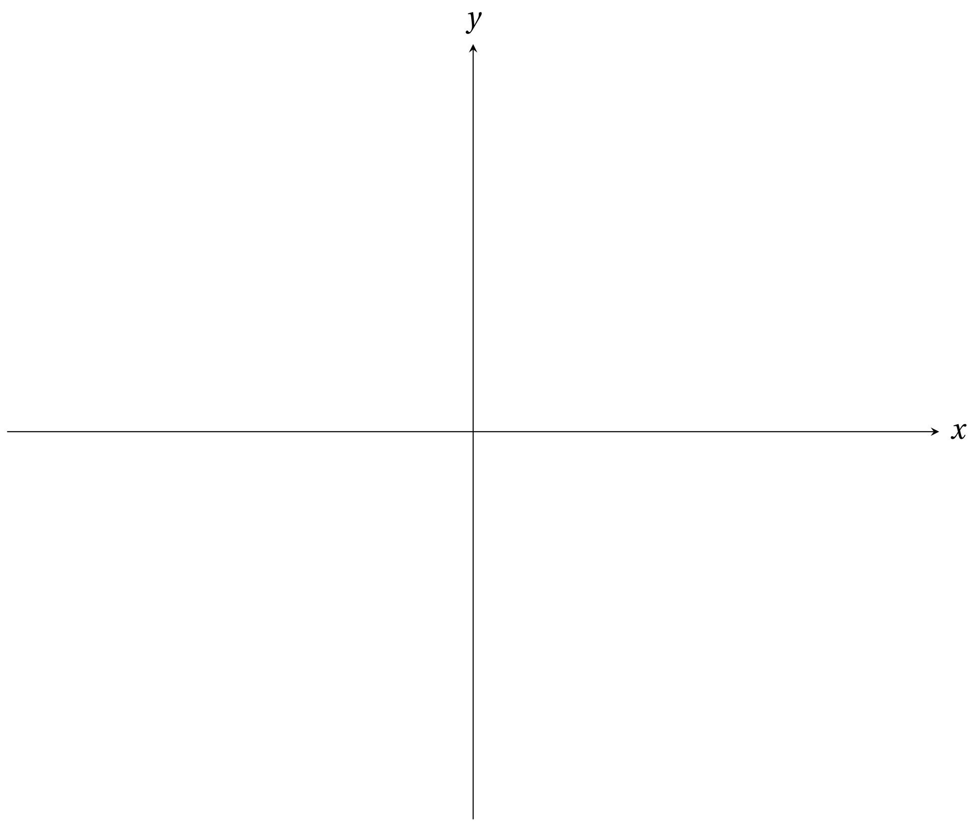
Hole(s):

x-intercept(s):

y-intercept:

VA:

End behaviour:



Example 2.34. Sketch the graph of $f(x) = \dfrac{x-1}{x+3}$.

Domain:

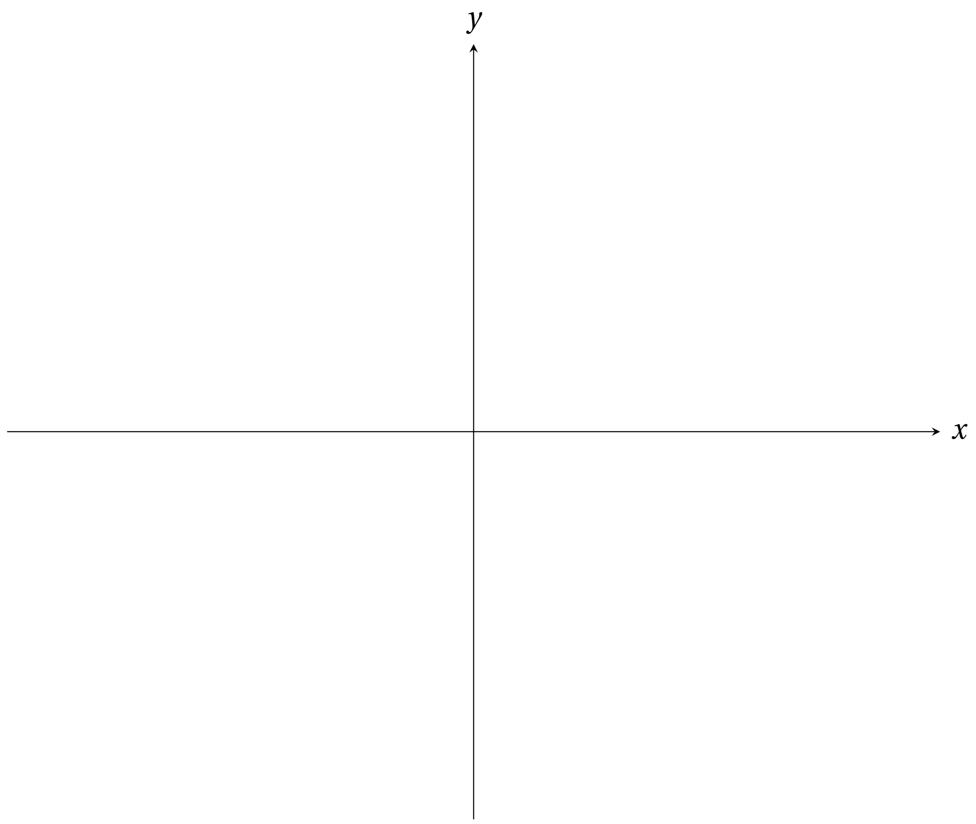
Hole(s):

x-intercept(s):

y-intercept:

VA:

End behaviour:



Example 2.35. Sketch the graph of $f(x) = \dfrac{2x+5}{4-x}$.

Domain:

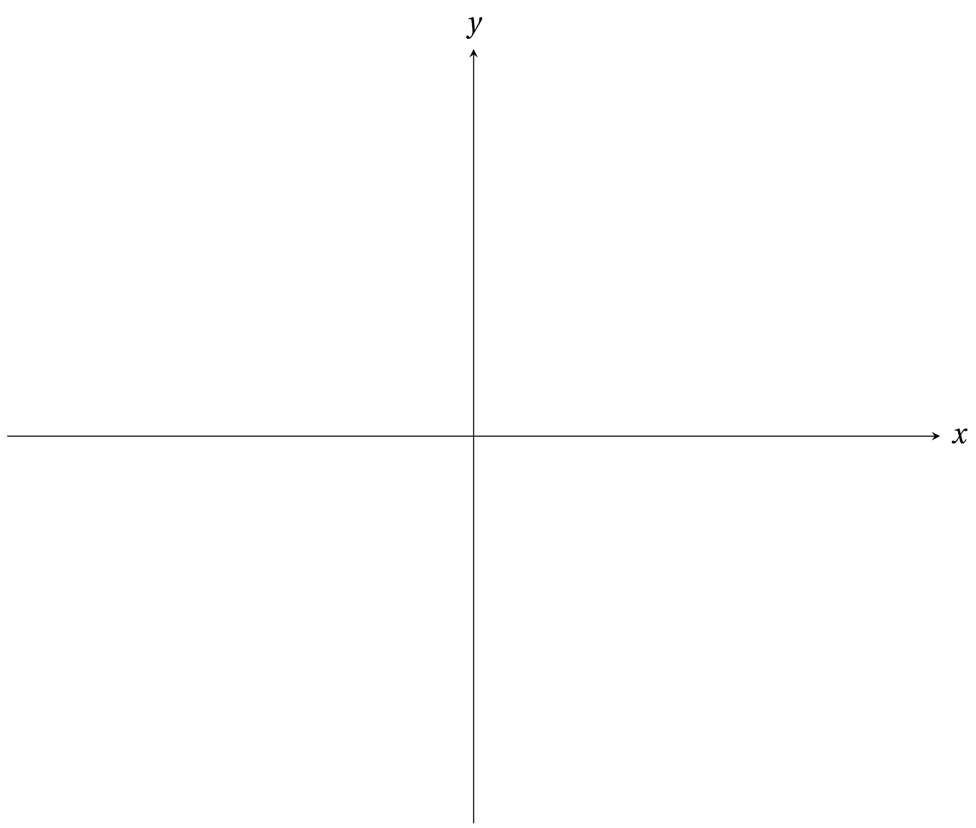
Hole(s):

x-intercept(s):

y-intercept:

VA:

End behaviour:



Example 2.36. Sketch the graph of $f(x) = \dfrac{x^2 - 2x - 8}{x^2 - 9}$.

Domain:

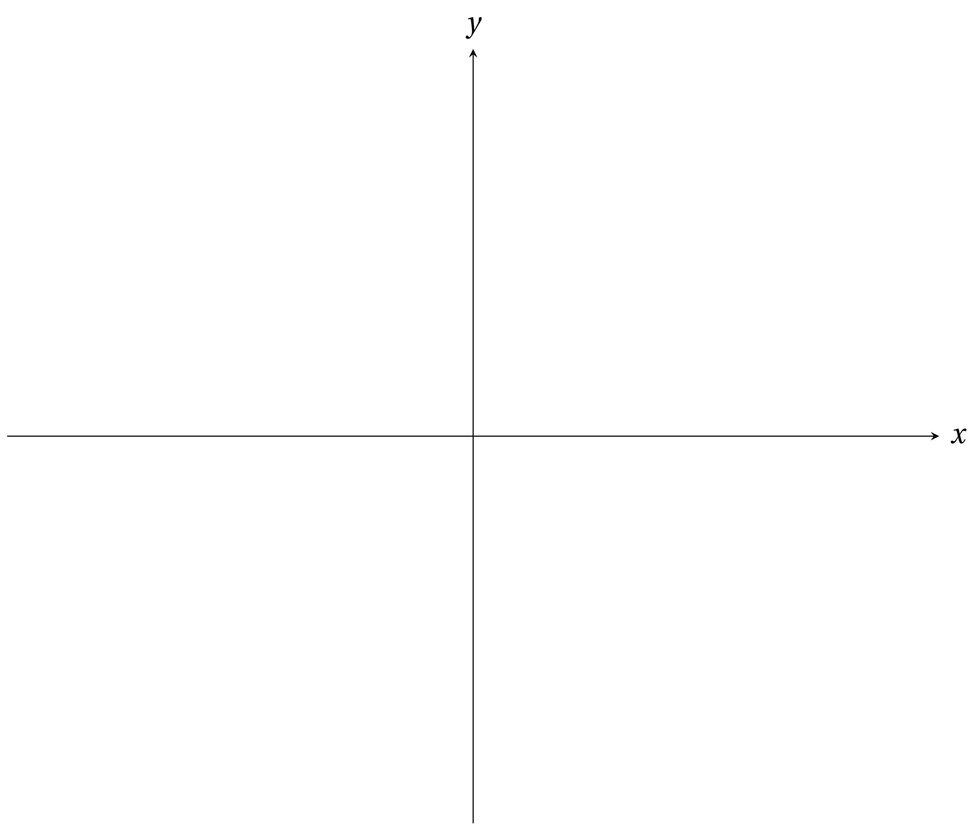
Hole(s):

x-intercept(s):

y-intercept:

VA:

End behaviour:



Example 2.37. Sketch the graph of $f(x) = \dfrac{x^2+2x-3}{x^2-6x+5}$.

Domain:

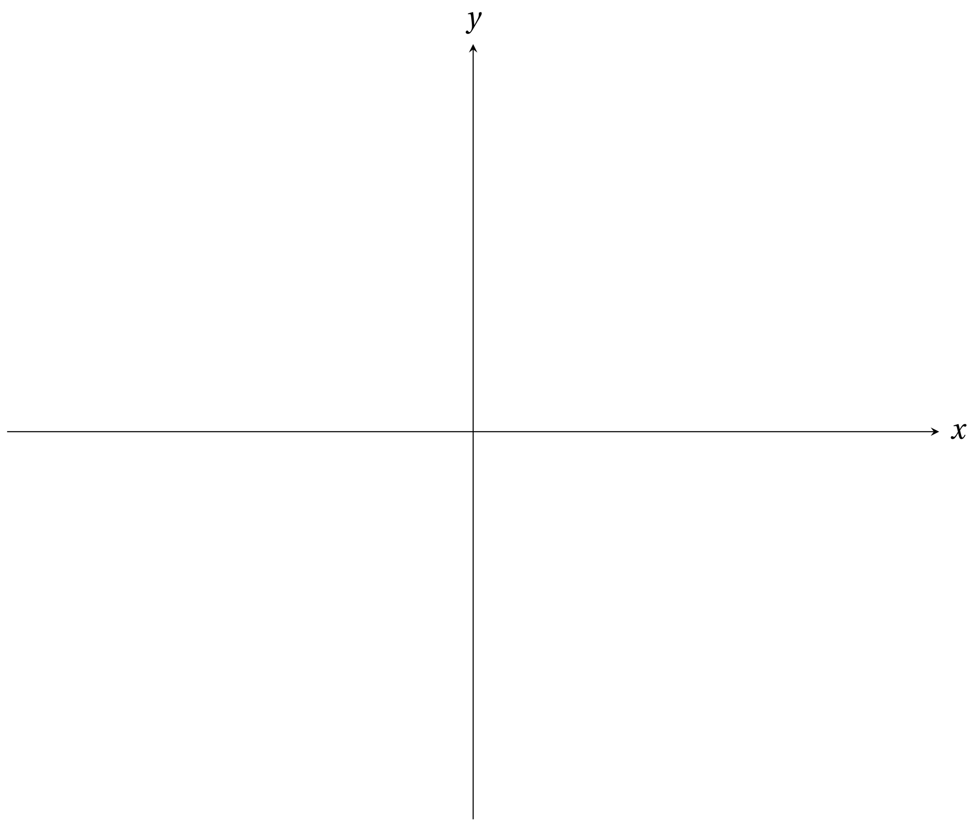
Hole(s):

x-intercept(s):

y-intercept:

VA:

End behaviour:



Example 2.38. Sketch the graph of $f(x) = \dfrac{2x^2-8}{x-1}$.

Domain:

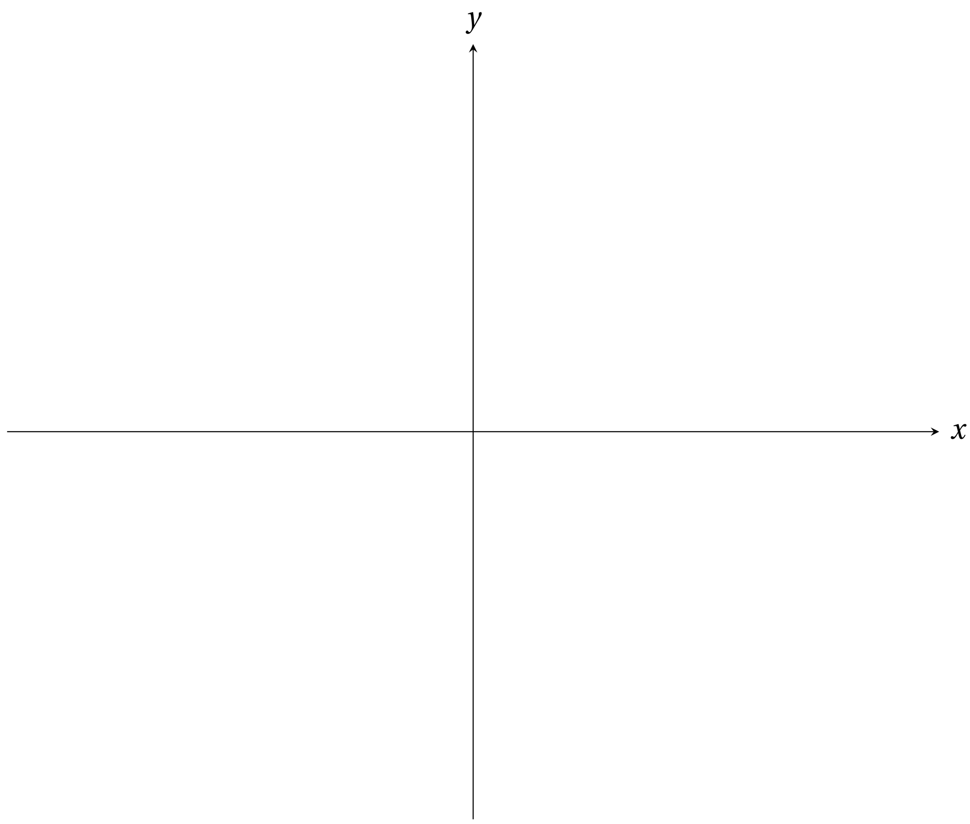
Hole(s):

x-intercept(s):

y-intercept:

VA:

End behaviour:



Example 2.39. Sketch the graph of $f(x) = \dfrac{x^2 - 2x}{x + 1}$.

Domain:

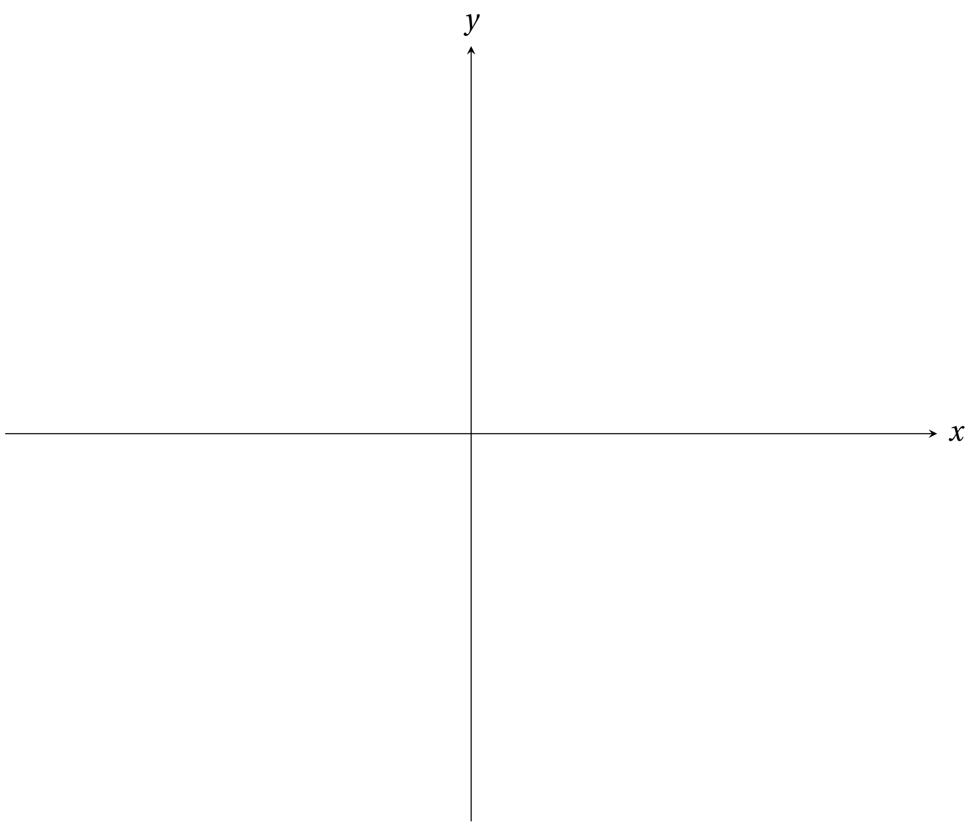
Hole(s):

x-intercept(s):

y-intercept:

VA:

End behaviour:



Example 2.40. Sketch the graph of $f(x) = \dfrac{x^2-4}{x-2}$.

Example 2.41. Sketch the graph of $f(x) = \dfrac{x^2+6x+8}{x+2}$.

2.7 Nonlinear Inequalities

Inequalities are basically equations but with equal sign replaced by one of these inequality signs $>, \geq, <, \leq$. There are two ways to solve inequalities: algebraically or graphically. But they are both similar in the sense that you need to factor certain equations first, this shouldn't come as a surprise for you because the very first step we do in graphing polynomials and rational functions is factoring to find their zeros.

We will focus on the graphical method, because we just spent a quarter of the semester learning how to graph equations. **You can easily solve inequalities just like I do, simply because you are able to graph equations by now, right?!** The set of all x-values that satisfy a given inequality is called a **solution set**. For polynomial and rational inequalities, the solution sets are intervals.

Polynomial inequalities

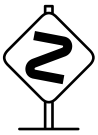
Example 2.42. Solve the following inequality.

(a) $x^2 + 6x + 5 \geq 0$

(b) $x^2 < 7x - 19$

(c) $x^3 \leq 3x^2 + 10x$

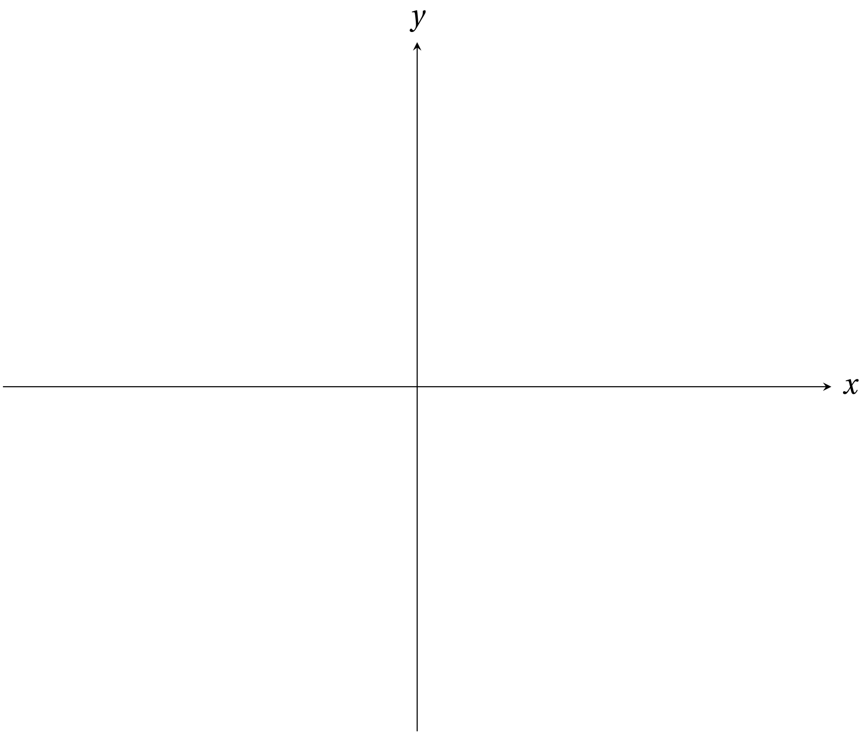
Rational inequalities



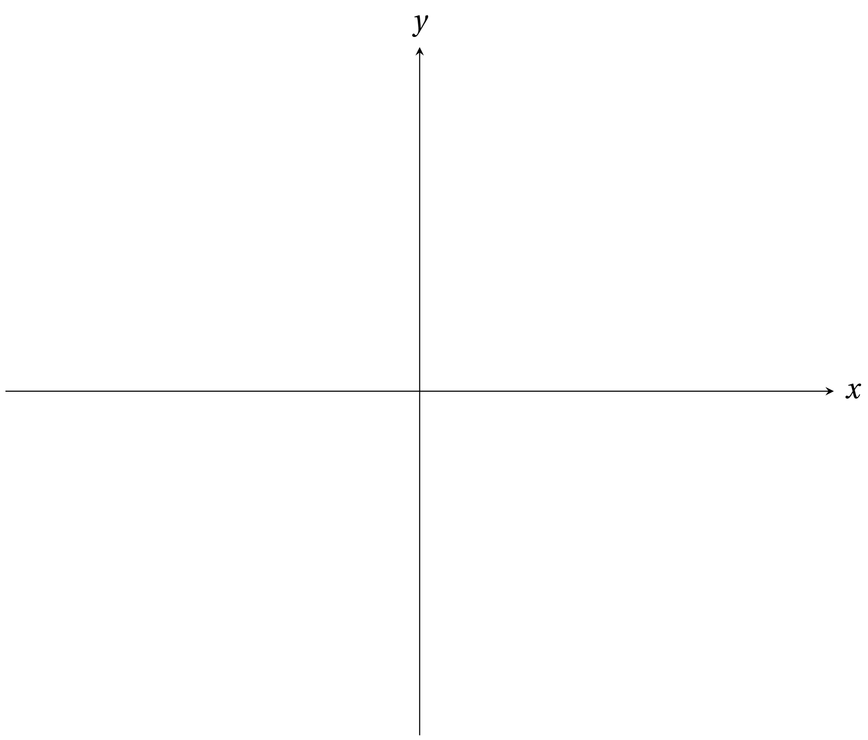
Don't even try to **CROSS-MULTIPLY** for rational inequalities.

Example 2.43. Solve $\dfrac{x+6}{x-1} < 2$.

METHOD I: Sketch and Solve



METHOD 2: Rearrange and Sketch



Example 2.44. Solve $\dfrac{x-2}{x+5} \geq 3$.

Example 2.45. Solve $\dfrac{2x+1}{x-7} \geq 3$.

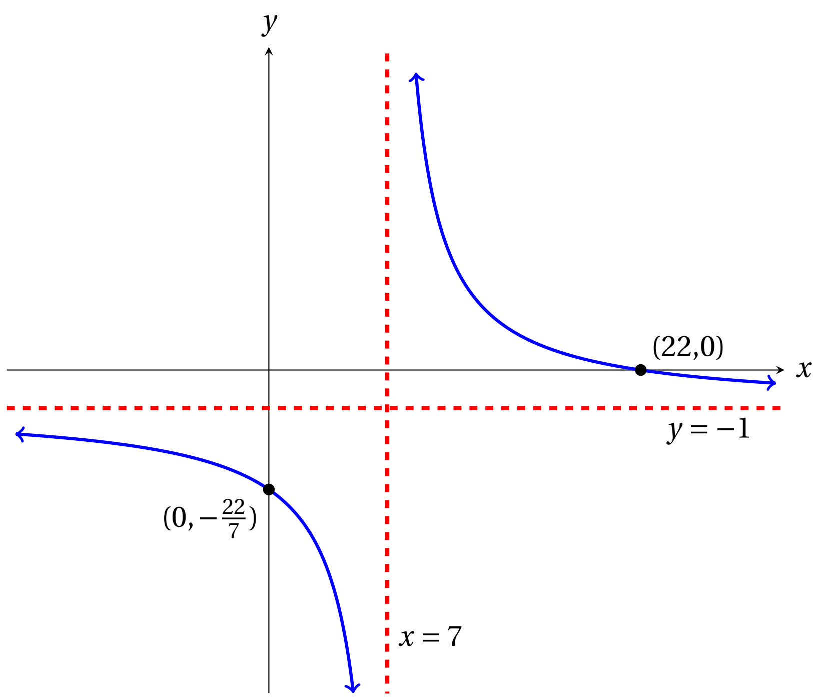
Let us solve by rearranging the inequality:

$$\begin{aligned}
\frac{2x+1}{x-7} - 3 &\geq 0 \\
\frac{2x+1}{x-7} - \frac{3(x-7)}{(x-7)} &\geq 0 \\
\frac{2x+1-3(x-7)}{x-7} &\geq 0 \\
\frac{2x+1-3x+21}{x-7} &\geq 0 \\
f(x) = \frac{-x+22}{x-7} &\geq 0
\end{aligned}$$

Since there is no cancelling factor, the graph does not have a hole and the VA is $x = 7$. Setting $x = 0$ gives the y-intercept $\left(0, \dfrac{22}{-7}\right) = \left(0, -\dfrac{22}{7}\right)$. To find the x-intercept, we set the numerator to be 0 and solve for x:

$$-x+22 = 0 \implies 22 = x.$$

Thus the x-intercept is $(22, 0)$. Looking at the ratio of the largest degree term in the numerator and denominator: $\dfrac{-x}{x} = -1$, we see that the graph has a horizonal asymptote $y = -1$. From the graph of $f(x)$, we see that the solution set is $(7, 22]$.



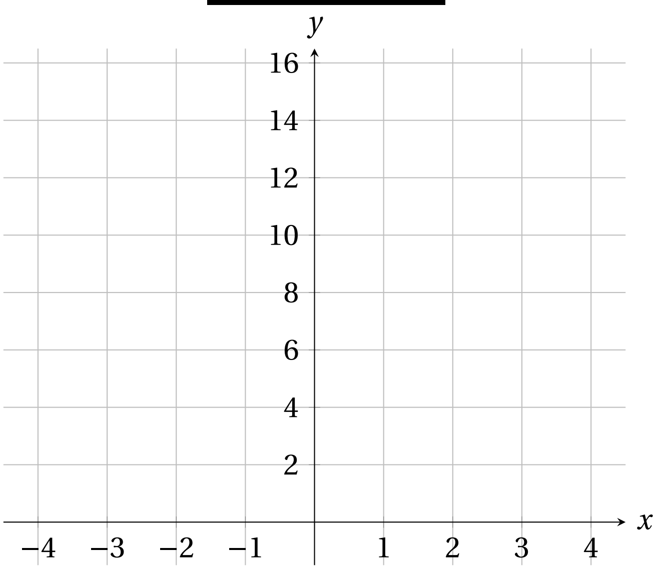
3 Exponential and Logarithmic Functions

3.1 Exponential Functions and Their Graphs

For $a > 0$ and $a \neq 1$, the function $f(x) = a^x$ is called the **exponential function with base *a***.

1. The domain of any exponential function is ________________.
2. The base $a = 1$ corresponds to the constant function $f(x) = 1^x \equiv 1$.
3. The base $a = e \approx 2.718281828$ is called the **natural base**. The function $f(x) = e^x$ is called the **natural exponential function**.

$\boxed{y = a^x \text{ for } a > 1}$



Domain :

Range :

x-intercept :

y-intercept:

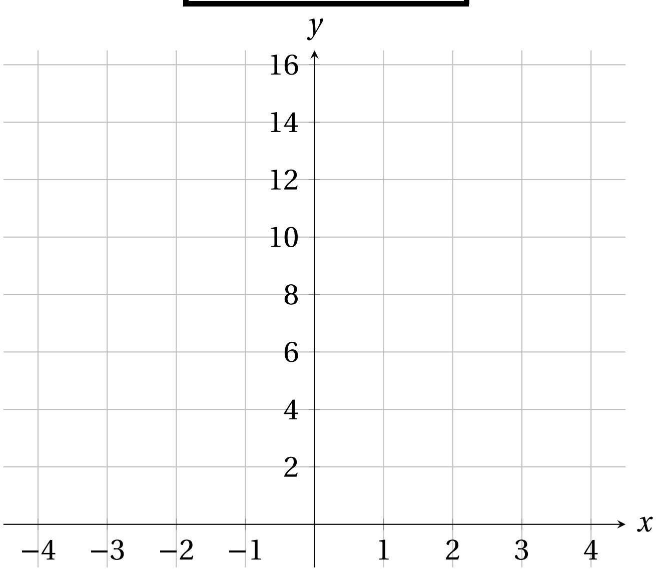
Increasing or Decreasing

Asymptote:

As $x \longrightarrow +\infty$, $f(x) \longrightarrow$ __________

As $x \longrightarrow -\infty$, $f(x) \longrightarrow$ __________

$\boxed{y = a^x \text{ for } 0 < a < 1}$



Domain :

Range :

x-intercept :

y-intercept:

Increasing or Decreasing

Asymptote:

As $x \longrightarrow +\infty$, $f(x) \longrightarrow$ __________

As $x \longrightarrow -\infty$, $f(x) \longrightarrow$ __________

It is useful to recall the **Rules for Exponents**.

♣ $a^m a^n =$

♣ $a^0 =$

♣ $\dfrac{a^m}{a^n} =$

♣ $a^{-n} =$

♣ $(a^m)^n =$

Since the graph of $f(x) = a^x$ passes the Horizontal Line Test, it satisfies the following **One-to-One Property** which can be used to solve simple exponential equations.

$$\boxed{a^x = a^y \iff x = y}$$

The equation must have the same base on each side before we can use the One-to-One Property!

Example 3.1. Solve for x in the following equation.

(a) $2^{x+4} = 2^6$

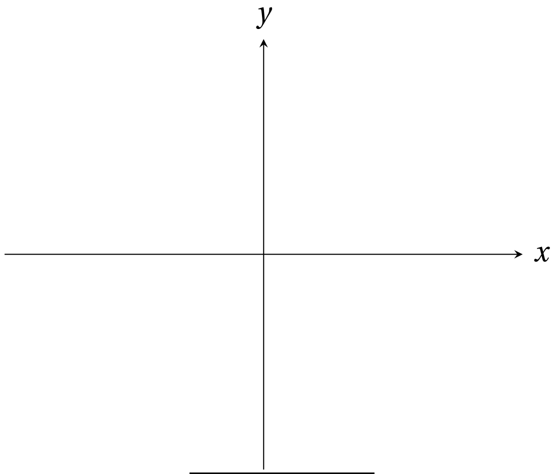
(b) $3^{2x-5} = \dfrac{1}{27}$

(c) $\left(\dfrac{1}{5}\right)^x = 125$

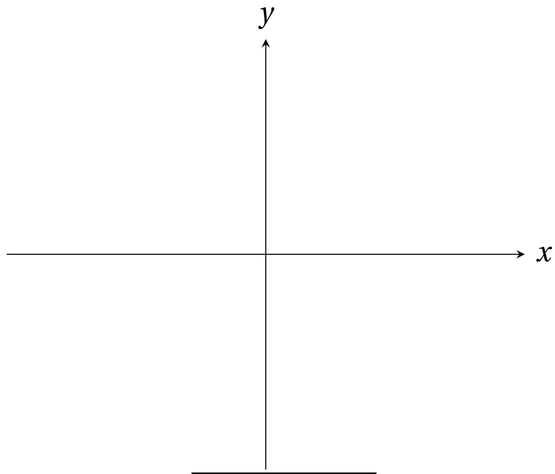
(d) $e^{x^2+3} = e^{12}$

Transformations of exponential functions

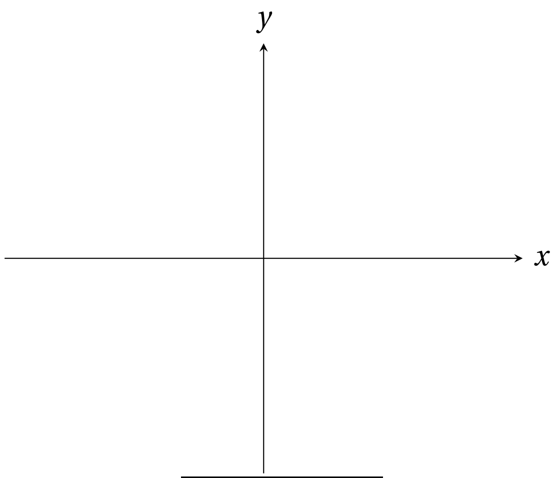
They are no different than the usual rules for sketching graphs of transformation of functions! Below are rigid transformations of the graph of $f(x) = 3^x$.



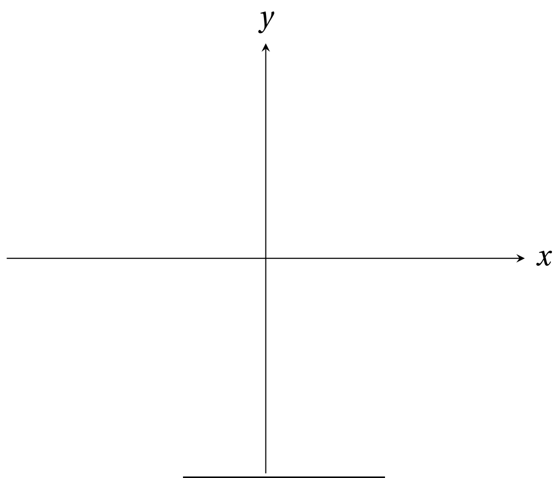
$f(x) = 3^{x+2}$



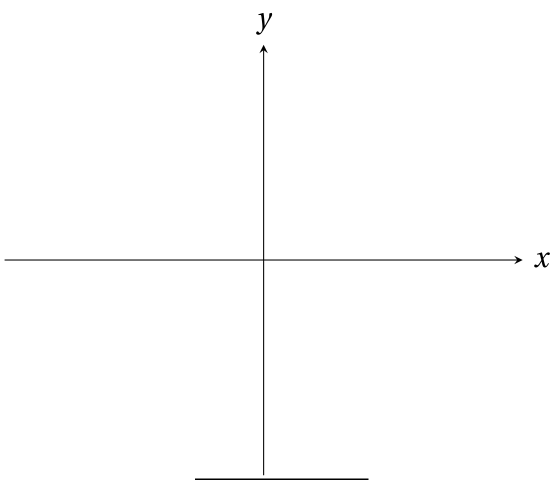
$f(x) = 3^{x-1}$



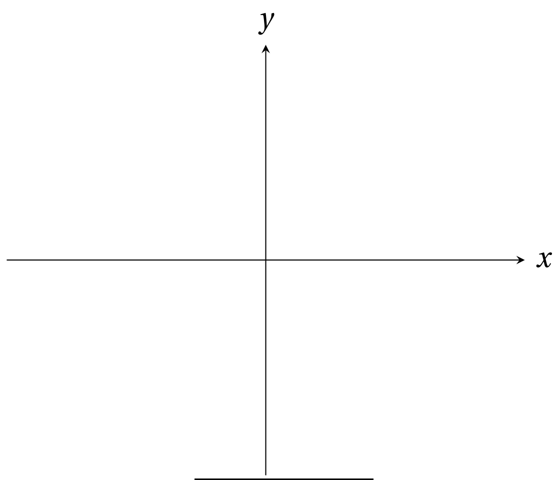
$f(x) = 3^x + 5$



$f(x) = 3^x - 4$



$f(x) = 3^{-x}$



$f(x) = -3^x$

Example 3.2. Find the domain, intercepts and asymptote of the function $f(x) = e^{x-1} + 3$. Sketch its graph and find the range.

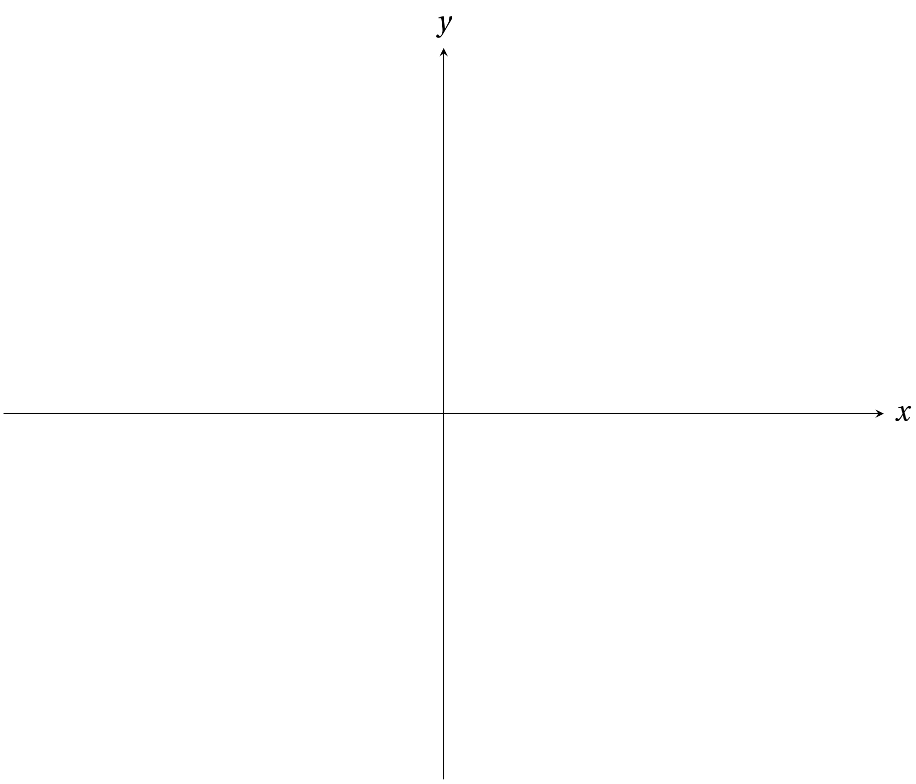
Domain:

Range:

x-intercept:

y-intercept:

Asymptote:



Example 3.3. Find the domain, intercepts and asymptote of the function $f(x) = 27 - 3^{x+2}$. Sketch its graph and find the range.

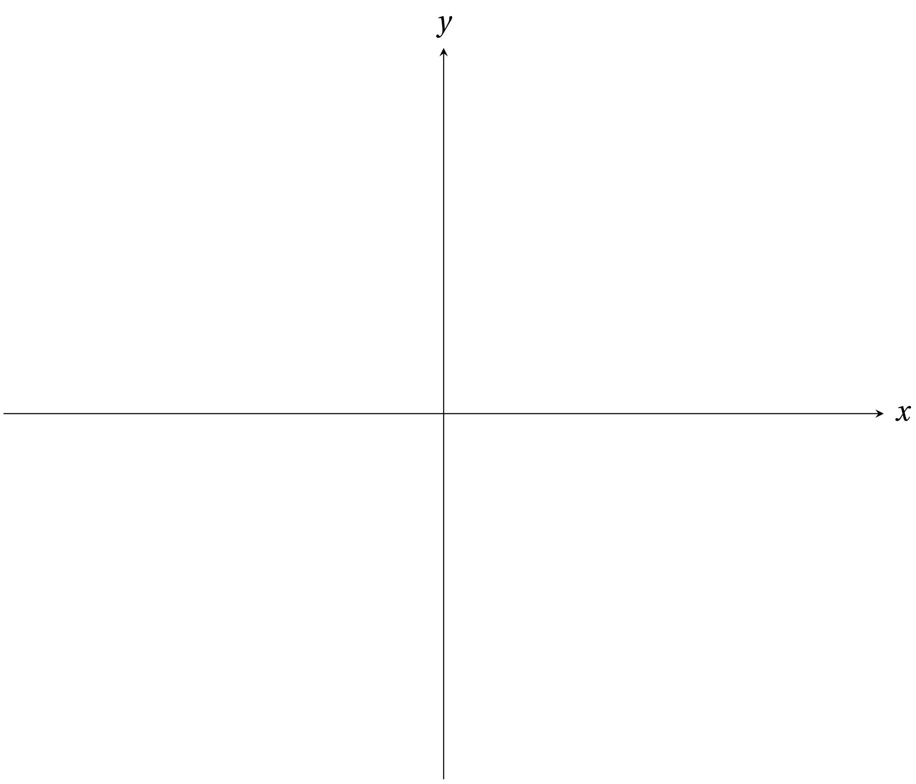
Domain:

Range:

x-intercept:

y-intercept:

Asymptote:



Applications

Exponential equations are applied in a wide variety of situations in science. We study two well-known examples: *an investment earning continuously compounded interest* and *population growth*. The model for **continuously compounded interest** is

$$
\begin{aligned}
A(t) = Pe^{rt}, \quad \text{where } P &= \text{initial deposit/investment/principal} \\
r &= \text{interest rate (in decimal form)} \\
t &= \text{time in years} \\
A(t) &= \text{amount of money after } t \text{ years.}
\end{aligned}
$$

Example 3.4. If you invest \$2000 at an annual interest rate of 3%, compounded continuously, how much will you have after 8 years?

Example 3.5 (Population Growth)**.** Suppose there is an annoying species of bug in the classroom that triples in population every hour. Given that there were approximately 100 bugs in the classroom, write down a model $B(t)$ for the number of population of this bug after t hours. How many bugs will there be after 6 hours?

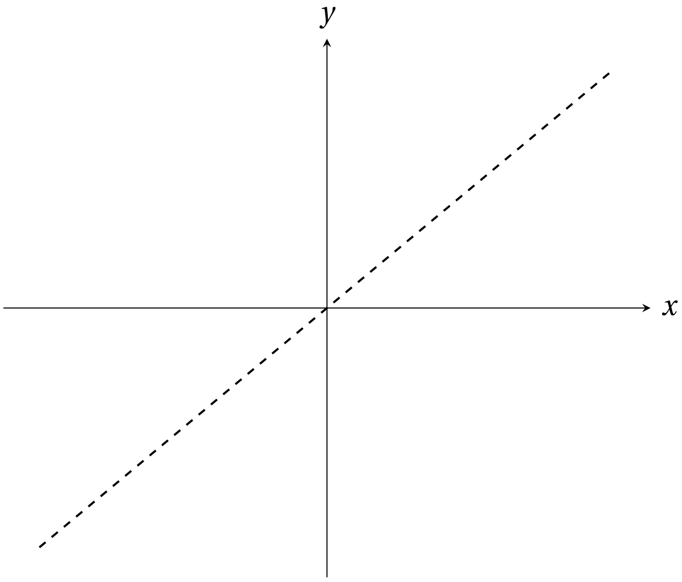
3.2 Logarithmic Functions and Their Graphs

Any exponential function has an inverse function since ______________________________.
This inverse function, denoted by $\log_a x$, is called the **logarithmic function with base *a***.

$$\boxed{y = \log_a x \quad \text{if and only if} \quad a^y = x}$$

- The function $\log_e x = \ln x$ is called the **natural logarithmic function**.
- The function $\log_{10} x = \log x$ is called the **common logarithmic function**.

$\boxed{y = \log_a x \text{ for } a > 1}$



Domain :

Range :

x-intercept :

y-intercept:

Increasing or Decreasing

Asymptote:

As $x \longrightarrow +\infty$, $f(x) \longrightarrow$ __________

As $x \longrightarrow -\infty$, $f(x) \longrightarrow$ __________

The logarithm is an exponent! Specifically, $\log_a x$ is the answer to the question "To what power must a be raised, in order to yield x?".

Example 3.6. Evaluate each of the following logarithms.

$\log_2 16 =$ BECAUSE

$\log_5 25 =$ BECAUSE

$\log_3 27 =$ BECAUSE

$\ln e =$ BECAUSE

Properties of logarithms

The following properties of the logarithmic function with base a follows directly from the definition.

- ♣ $\log_a 1 = 0$ BECAUSE

- ♣ $\log_a a = 1$ BECAUSE

- ♣ **[Inverse Property]** $\log_a a^x = x$ for all real numbers x.
- ♣ **[Inverse Property]** $a^{\log_a x} = x$ for all $x > 0$.
- ♣ **[One-to-One Property]** $\log_a x = \log_a y \iff x = y$.

Example 3.7. Simplify the following expressions.

(a) $\log_7 1 =$

(b) $\log_{2018} 2018 =$

(c) $\log_6 6^{-4} =$

(d) $5^{\log_5 3} =$

(e) $\ln 1 =$

(f) $\ln e =$

(g) $\ln e^{-2.5} =$

(h) $e^{\ln 7} =$

Example 3.8. Use the One-to-One Property to solve for x in the following equations.

(a) $\log_3(2x-4) = \log_3(7-x)$

(b) $\ln(x^2 - 5x) = \ln(4x+10)$

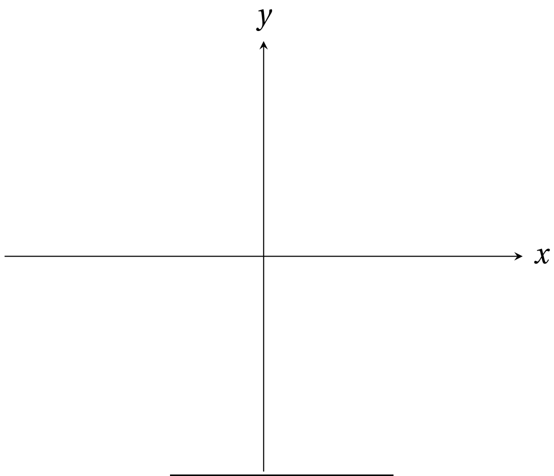
Inverting exponential and logarithmic equations

In solving exponential and logarithmic equations, if the One-to-One Property is not applicable, then the crucial step is to be able to write the exponential equation in logarithmic form and vice versa; see Section 3.4. To this end, recall the definition of logarithmic function as the inverse of exponential function:

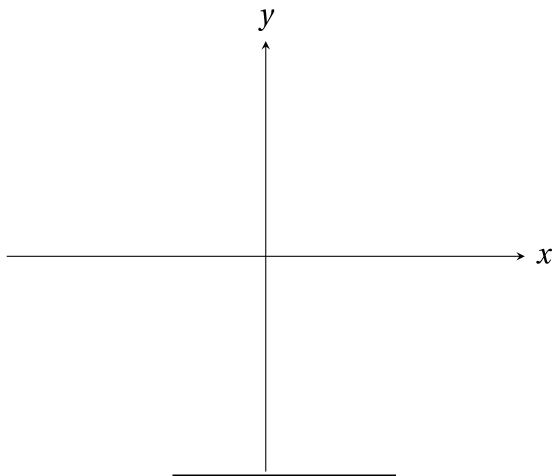
- *Writing exponential equations in logarithmic form* means *taking the logarithm of each side*. This is the same as saying **"invert the operation of exponentiation"**. For example,
  - ♡ $2^{3x-1} = 11$ is the same as $\log_2\left(2^{3x-1}\right) = \log_2(11)$, or $3x - 1 = \log_2(11)$.
  - ♡ $e^{6-2x} = 5$ is the same as
  - ♡ $\left(\frac{1}{5}\right)^{-4x+3} = 2$ is the same as
  - ♡ $7^{2-3x} = 13$ is the same as
  - ♡ $9^{1/2} = 3$ is the same as
- *Writing logarithmic equations in exponential form* means *exponentiate each side*. This is the same as saying **"invert the operation of taking logarithm"**. For example,
  - ♡ $\log_2 x = -3$ is the same as $2^{\log_2 x} = 2^{-3}$, or $x = 2^{-3} = \frac{1}{8}$.
  - ♡ $\ln(3x - 7) = 5$ is the same as
  - ♡ $\log_7(3 - 2x) = 12$ is the same as
  - ♡ $\log_{\frac{1}{5}}(x^2 - 3) = -6$ is the same as
  - ♡ $\log_3 81 = 4$ is the same as

Transformations of logarithmic functions

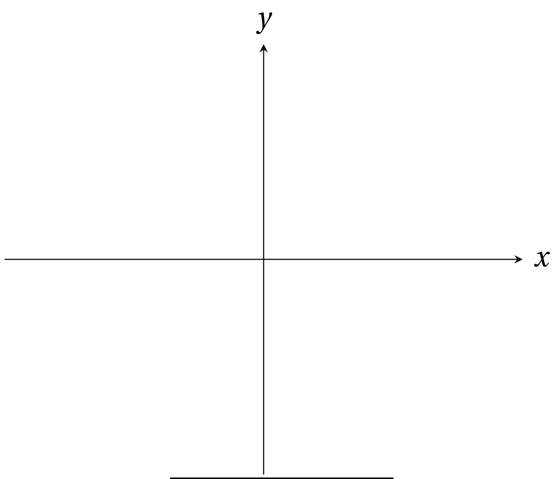
Again, these are no different than the usual rules for sketching graphs of transformation of functions! Below are rigid transformations of the graph of $f(x) = \log_2 x$.



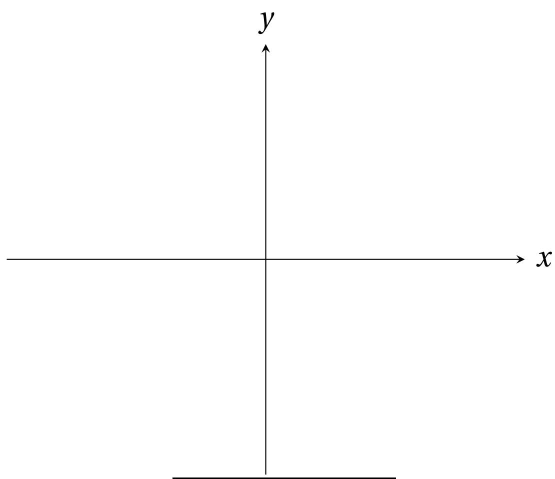
$y = \log_2(x-3)$



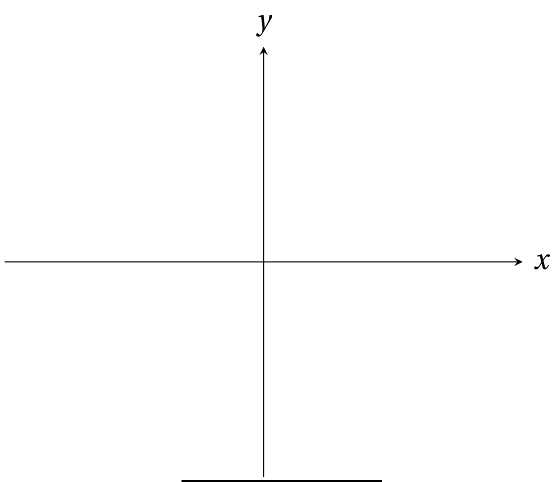
$y = \log_2(x+7)$



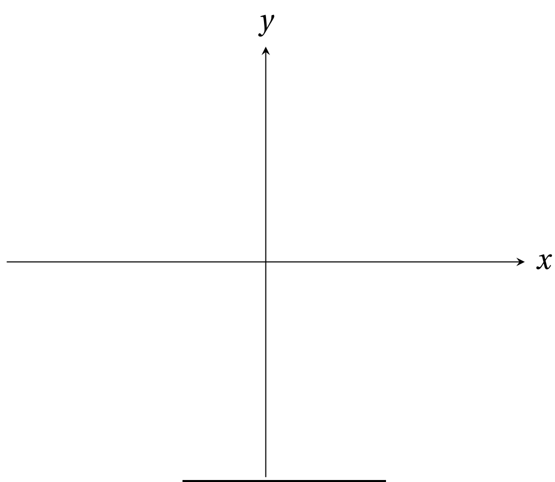
$y = \log_2(x) + 1$



$y = \log_2(x) - 3$



$y = \log_2(-x)$



$y = -\log_2(x)$

To find the domain of logarithmic functions, we exploit the fact that **logarithmic functions can only take positive inputs** because of the vertical asymptote. Knowing the domain is useful in sketching graphs of transformations of logarithmic functions!

Example 3.9. Given the logarithmic function $f(x) = -\ln(x-5)$, find its domain, range, intercepts and asymptote. Sketch its graph.

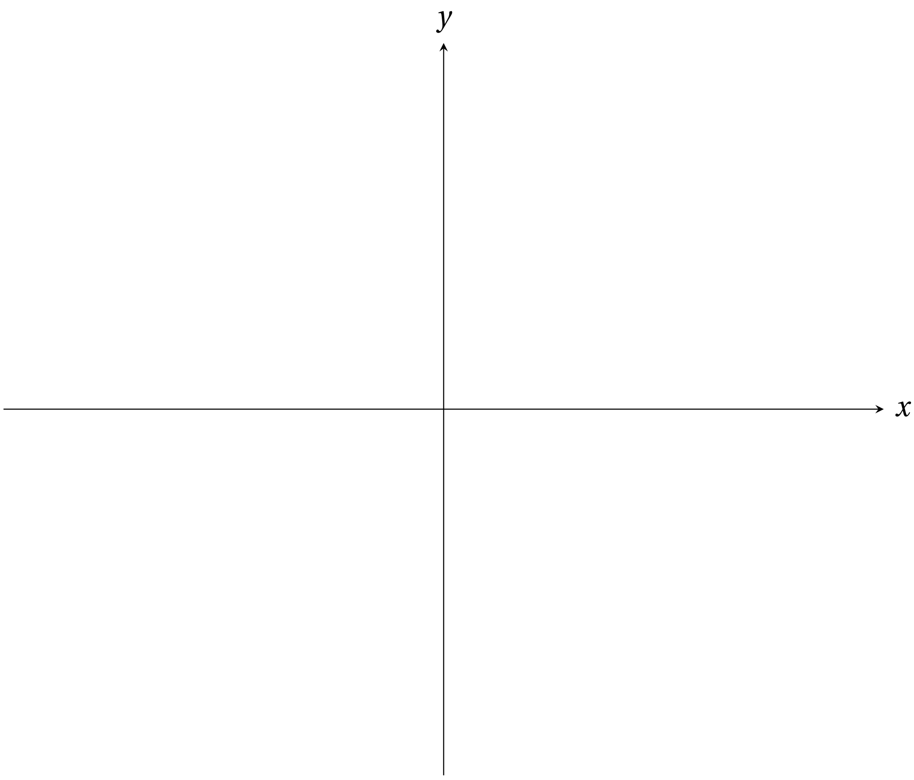
Domain:

Range:

x-intercept:

y-intercept:

Asymptote:



Example 3.10. Given the logarithmic function $f(x) = \log_2(4-x)+3$, find its domain, range, intercepts and asymptote. Sketch its graph.

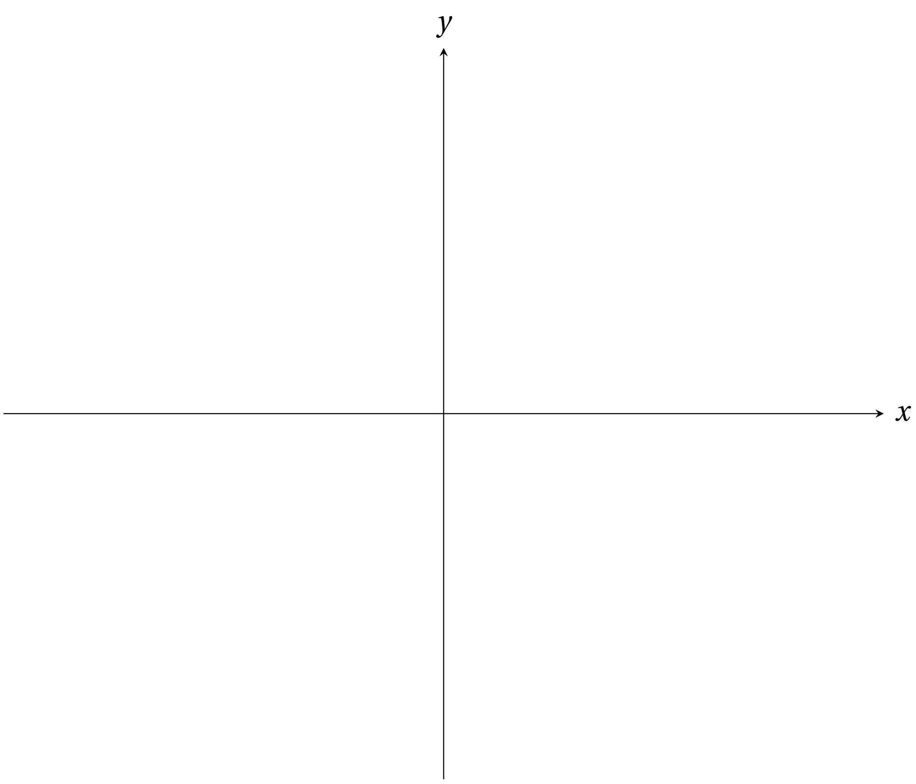
Domain:

Range:

x-intercept:

y-intercept:

Asymptote:



3.3 Properties of Logarithms

A lot of students abuse the properties of exponents without really understand them. Because the logarithm is the inverse operation to exponentiation, properties of logarithms follows immediately if one truly comprehends the properties of exponents but some might disagree. Buckle your seatbelt, as you are about to see one of the greatest, if not the greatest revelation of your mathematical journey.

Logarithms turn multiplication into addition.

Change of bases

This relates logarithms with common logarithms $\log(\cdot)$ and natural logarithms $\ln(\cdot)$ and so we can evaluate (using scientific calculator) logarithms of any bases. Let us try to write the following two logarithms as a ratio of common logarithms and natural logarithms.

$\log_8 21$

(a) Common logarithms

(b) Natural logarithms

$\log_{16} 5$

(a) Common logarithms

(b) Natural logarithms

Finding exact values

Find the exact value using only the properties of logarithms.

(a) $\log_3 \dfrac{1}{27}$

(b) $\log_2 \sqrt{8}$

(c) $\log_{\frac{1}{5}}(125)$

(d) $3\ln e^4 + 2\ln e^2$

(e) $\log_4 128 - \log_4 2$

(f) $\log_6 54 - \log_6 3 + \log_6 2$

Expanding logarithmic expressions

(a) $\ln\left(x^4\sqrt{y}z^3\right)$

(b) $\log_9\left(\dfrac{2x^3z}{y-7}\right)$

(c) $\ln\left(\dfrac{z^2-8}{\sqrt{x^3}y^{10}}\right)$

Solution:

$$\begin{aligned}
\ln\left(\frac{z^2-8}{\sqrt{x^3}y^{10}}\right) &= \ln(z^2-8)-\ln\left(\sqrt{x^3}y^{10}\right) && \left[\text{Quotient rule.}\right]\\
&= \ln(z^2-8)-\ln\left(x^{3/2}y^{10}\right) && \left[\text{Using } \sqrt{a}=a^{1/2} \text{ and } (a^m)^n=a^{mn}.\right]\\
&= \ln(z^2-8)-\left[\ln x^{3/2}+\ln y^{10}\right] && \left[\text{Product rule.}\right]\\
&= \ln(z^2-8)-\left[\frac{3}{2}\ln(x)+10\ln y\right] && \left[\text{Power rule.}\right]\\
&= \ln(z^2-8)-\frac{3}{2}\ln(x)-10\ln y.
\end{aligned}$$

Condensing logarithmic expressions

Condense or simplify the following expressions to the logarithm of a single quantity.

(a) $4\ln(x-5)+2\ln(x)-3\ln(x^2-1)$

(b) $7\log_5(x)-4\log_5(z)+\dfrac{3}{2}\log_5 y^2$

(c) $5\ln(x+2)+\dfrac{1}{3}\ln(z^2)-4\ln(y-1)$

Solution:

$$\begin{aligned}
5\ln(x+2)+\frac{1}{3}\ln(z^2)-4\ln(y-1) &= \ln(x+2)^5+\ln(z^{2/3})-\ln(y-1)^4 && \text{[Power rule.]}\\
&= \ln\left[(x+2)^5 z^{2/3}\right]-\ln(y-1)^4 && \text{[Product rule.]}\\
&= \ln\left[\frac{(x+2)^5 z^{2/3}}{(y-1)^4}\right] && \text{[Quotient rule.]}
\end{aligned}$$

3.4 Exponential and Logarithmic Equations

We have seen how to solve exponential and logarithmic equations using One-to-One property, but this is applicable if and only if all the terms have the same base. Otherwise, we have to invert the operation.

Solving exponential equations

We first **condense** (simplify) the given exponential equation. Then either we convert each side to have the same base so that we can apply One-to-One property, or we take the logarithm of each side.

Example 3.11. Solve for x in the following exponential equations.

(a) $\left(\frac{1}{9}\right)^x = 27^{2-x}$

(b) $3(2^{5x-1}) - 7 = 41$

(c) $4(e^{6-x}) + 5 = 12$

(d) $3^{2x+3} = 8\left(3^{1-x}\right)$

(e) $\dfrac{400}{1+e^{-x}} = 350$

(f) **[Quadratic type]** $e^{2x} - 4e^{x} - 5 = 0$

Solving logarithmic equations

We first **condense** (simplify) the given logarithmic equation. Then either we convert each side to have same base so that we can apply One-to-One property, or we exponentiate each side. Because the domain of a logarithmic function is generally not all real numbers, we need to check for **extraneous solutions**, *i.e.* solutions that appear to be valid but do not satisfy the original (unsimplified) equation.

Example 3.12. Solve for x in the following logarithmic equations.

(a) $2 + 3\ln x = 12$

(b) $4\log_2(3x-1) + 11 = -5$

(c) $\log_8(3x-17)-\log_8(8-x)=\dfrac{2}{3}$

(d) $\ln(x-4)+\ln(x+1)=\ln 6$

(e) $\log_3 x + \log_3(x+6) = 3$

(f) $\log(x+12) - \log(x-3) = \log x$

3.5 Exponential Models

The key to successfully solving exponential model problems is to be able to convert the given information into equations.

Continuously compounded interest

The model for **continuously compounded interest** is

$$A(t) = Pe^{rt}, \quad \text{where} \quad \begin{aligned} P &= \text{initial deposit/investment/principal} \\ r &= \text{interest rate (in decimal form)} \\ t &= \text{time in years} \\ A(t) &= \text{amount of money after } t \text{ years.} \end{aligned}$$

Example 3.13. If you invest \$8000 at an annual interest rate of 5% compounded continuously,

(a) how much will you have after 7 years?

(b) how long does it take to double the investment?

Exponential growth model

The **population (exponential) growth model** is

$$A(t) = Pe^{rt}, \quad \text{where} \quad \begin{aligned} P &= \text{initial population} \\ r &= \text{growth rate} \\ t &= \text{time in hours/days/weeks (specified in the problem)} \\ A(t) &= \text{number of population at time } t. \end{aligned}$$

We point out that this model is applicable to any exponential growth problems, but the unit of time has to be adjusted accordingly.

Example 3.14. The number of bacteria in a culture is increasing according to the law of exponential growth. The initial population is 300 bacteria and the population after 10 hours is double the population after one hour. How many bacteria will there be after 6 hours?

Example 3.15. The population of fruit flies is increasing according to the law of exponential growth. After 2 days there are 250 flies, and after 5 days there are 700 flies. Find the growth rate r and the population of fruit flies after 1 week.

Exponential decay model

The **radioactive decay model** is

$$A(t) = Pe^{rt}, \quad \text{where} \quad \begin{aligned} P &= \text{initial quantity of isotopes} \\ r &= \text{decay rate} \\ t &= \text{time in years} \\ A(t) &= \text{amount of isotopes after } t \text{ years.} \end{aligned}$$

We point out that this model is applicable for any exponential decay problems, then the unit of t has to be adjusted accordingly.

Archaeologists are interested in the **half-life**, which we denote by T_{half}. It is **the time taken to halve the initial quantity of radioactive isotope**. We may derive an explicit formula that relates the half-life T_{half} and the decay rate $r < 0$. By definition, we have

$$\begin{aligned} Pe^{rT_{\text{half}}} &= \frac{P}{2} \\ e^{rT_{\text{half}}} &= \frac{1}{2} && \left[\text{Divide each side by } P \neq 0.\right] \\ rT_{\text{half}} &= \ln\left(\frac{1}{2}\right) && \left[\text{Take ln of each side.}\right] \\ rT_{\text{half}} &= \ln 1 - \ln 2 = -\ln 2 && \left[\text{Quotient rule.}\right] \\ T_{\text{half}} &= -\frac{\ln 2}{r}. \end{aligned}$$

Example 3.16. Suppose we have 10g of carbon 14 isotope ^{14}C and it begins to decay with a half-life of about 5700 years.

(a) Find the amount of ^{14}C after 500 years.

(b) How many years will it take to decay to 50% of its initial amount?

Example 3.17. Suppose a drug has a half-life of 4 hours.

(a) How many hours will it take to decay to 25% of its initial amount?

(b) How many hours will it take to decay to 60% of its initial amount?

7 Systems of Equations and Inequalities

7.1 Linear and Nonlinear Systems of Equations

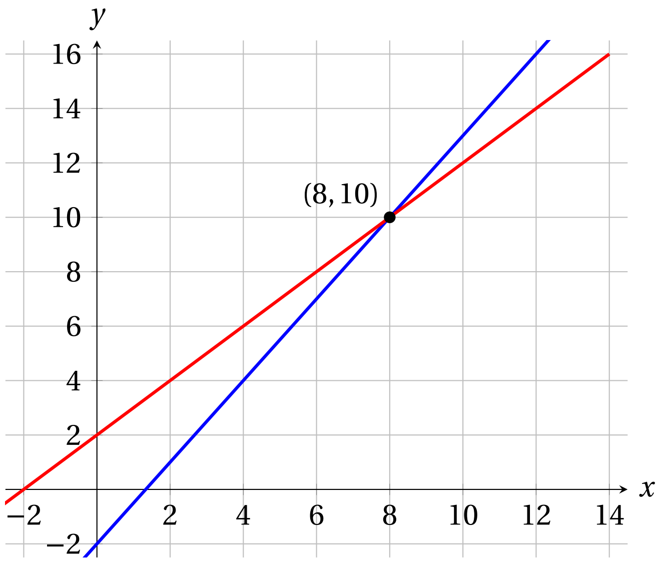
Many problems in science, business and engineering involve two or more equations in two or more variables. These equations constitute a **system of equations** and solving the system of equations means finding all possible ordered pairs, called **solutions**, such that each ordered pair satisfies the given system of equations. Consider solving the following 2×2 linear system, consisting of 2 linear equations with 2 variables x and y:

$$\begin{cases} 3x - 2y = 4 \\ x - y = -2 \end{cases} \tag{7.1}$$

We claim that $(x, y) = (8, 10)$ is a solution. This can be checked by simply substituting the point $(8, 10)$ into the system (7.1):

$3x - 2y =$

$x - y =$



Moreover, we can clearly see from the graph that $(8, 10)$ is the only solution!

Method of substitution (two-variable systems)

1. Choose one of the given two equations. For the chosen equation, choose a variable and solve (write) the chosen variable in terms of the other variable.
2. Substitute the expression found in Step 1 into the other equation. This results in a single equation with only one variable, which we can now solve!
3. *Back-substitute* the value obtained in Step 2 into the expression obtained in Step 1 to find the value of the other variable.
4. Write the solution in ordered pairs.

Example 7.1. Solve the following systems of linear equations using method of substitution.

(a)

$$\begin{cases} -x + 2y = 11 \\ x - y = -8 \end{cases}$$

(b)

$$\begin{cases} 5x - 2y = 1 \\ -x + y = 4 \end{cases}$$

(c)

$$\begin{cases} x - 2y = 2 \\ 3x + y = 20 \end{cases}$$

Example 7.2. Solve the following systems of nonlinear equations using method of substitution.

(a)

$$\begin{cases} x + y = 4 \\ x^2 + y^2 - 4x = 0 \end{cases}$$

(b)

$$\begin{cases} -2x + y = -3 \\ x^2 + y = 5 \end{cases}$$

(c)

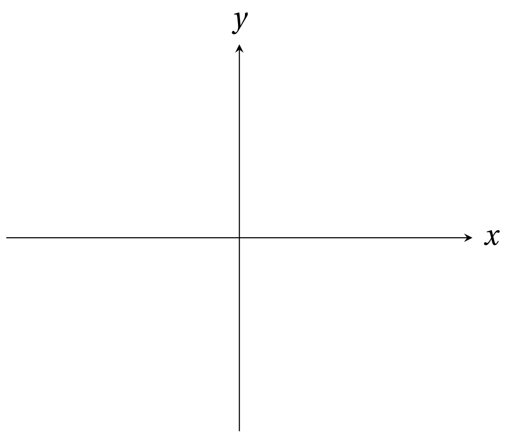
$$\begin{cases} y^2 - 18 = 2x \\ 3 + x = 2y \end{cases}$$

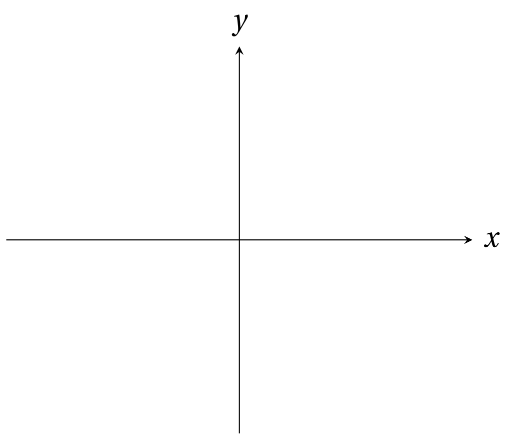
7.2 Two-Variable Linear Systems

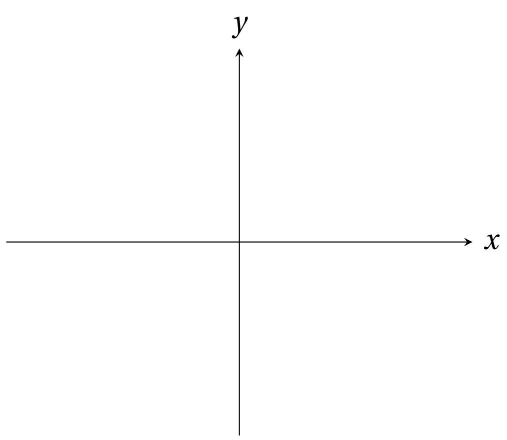
In addition to method of substitution, we also study the **method of elimination** for solving linear systems of two variables. The crucial step is to **add or substract "suitable multiples" of equations to eliminate one of the variables.** Before we demonstrate how this method works, let us first explore the geometric interpretation of solutions of linear systems.

Graphical interpretation of solutions

Using the fact that the graph of a linear equation (in two variables) is a line, there are only three possibilities for the number of solutions for a 2×2 linear system.







Example 7.3. Solve the following systems of linear equations using method of elimination.

(a)

$$\begin{cases} 2x + y = -24 \\ 2x - y = 4 \end{cases}$$

(b)

$$\begin{cases} 2x + 3y = 8 \\ 4x - y = -12 \end{cases}$$

(c)

$$\begin{cases} -4x + 3y = 2 \\ 5x - 6y = -7 \end{cases}$$

(d)

$$\begin{cases} 3x + 2y = 7 \\ 2x + 5y = 1 \end{cases}$$

(e)

$$\begin{cases} 2x - y = 1 \\ 4x - 2y = 2 \end{cases}$$

(f)

$$\begin{cases} 2x + 6y = 7 \\ x + 3y = 3 \end{cases}$$

7.3 Multivariable Linear Systems

Actually, the method of elimination can also be applied to $n \times n$ linear systems with $n \geq 3$. Consider the following relatively simple 3×3 linear system.

$$\begin{cases} 2x - y + 5z = 22 \\ \qquad\; y + 3z = \;\;6 \\ \qquad\qquad 4z = 12 \end{cases}$$

Observe that

- ♣ we can solve for z in equation 3;
- ♣ from equation 2, we can solve for y once z is known;
- ♣ from equation 1, we can solve for x once z and y are known.

This is the so called **back-substitution** method:

$$4z = 12 \implies$$

$$y + 3z = 6 \implies$$

$$2x - y + 5z = 22 \implies$$

Example 7.4. Use back-substitution to solve the following system of linear equations.

$$\begin{cases} x - \;\; y + \;\; z = \;\;3 \\ \quad\; 3y + 4z = -1 \\ \qquad\; -2z = \;\;5 \end{cases}$$

Gaussian elimination

The systematic way of applying the method of elimination is called *Gaussian elimination.* The main idea is to reduce a given system of linear equations into **row-echelon form**, which means that it has a "stair-step" pattern with diagonals 1; then we can simply use back-substitution to solve the system. There are three operations of Gaussian elimination.

1. Interchange/swap two equations.
2. Multiply or divide one of the equations by a nonzero constant.
3. Add (subtract) a multiple of one equation to (from) another equation to replace the latter equation.

Example 7.5. Use Gaussian elimination to solve the following 2×2 linear system.

$$\begin{cases} x + 2y = 1 \\ 2x - y = 7 \end{cases}$$

Example 7.6. Use Gaussian elimination to solve the following 2×2 linear system.

$$\begin{cases} 2x - 6y = 2 \\ -3x + 11y = 1 \end{cases}$$

Remark 7.7. Similar to a 2×2 linear system, there are only 3 possible scenarios for the number of solutions of a linear system, and only one of the following can be true.

1. There is exactly one solution. [Consistent]

2. There are infinitely many solutions. [Consistent]

3. There is no solution. [Inconsistent]

Example 7.8 (Exactly one solution)**.** Solve the following 3×3 linear systems.

(a)

$$\begin{cases} x + y + z = 3 \\ 2x + 3y + 4z = 5 \\ 3x + 5y + 4z = 13 \end{cases}$$

(b)

$$\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ 3x + y - z = 2 \end{cases}$$

(c)

$$\begin{cases} 2x - 2y - 6z = -4 \\ -3x + 2y + 6z = 1 \\ x - y - 5z = -3 \end{cases}$$

(d)

$$\begin{cases} x \phantom{{}-y} + 2z = 5 \\ 3x - y - z = 1 \\ 6x - y + 4z = 24 \end{cases}$$

Example 7.9 (Infinitely many solutions)**.** Solve the following 3×3 linear system.

$$\begin{cases} x + 2y - z = -4 \\ 2x + 3y + z = 0 \\ 3x + 7y - 6z = -20 \end{cases}$$

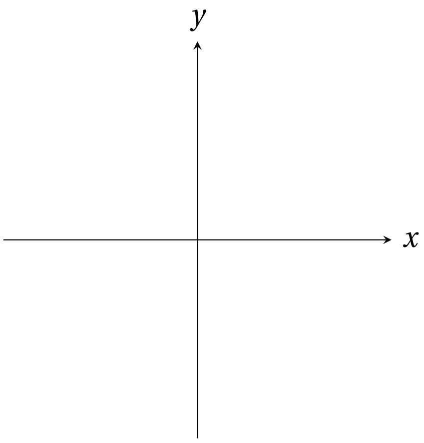
Example 7.10 (No solution)**.** Solve the following 3×3 linear system.

$$\begin{cases} x + 4z = 1 \\ x + y + 10z = 10 \\ 2x - y + 2z = -5 \end{cases}$$

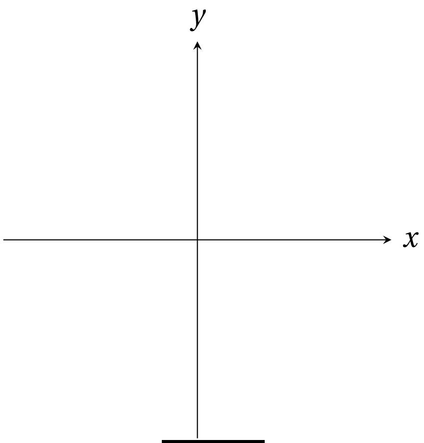
7.5 Systems of Inequalities

After many weeks of teaching and bonding and criticising and joking and what not, if you think that I am just going to let you off the hook for graphing functions, then you are in for a treat ((((((((: The goal of this final section is to graphically solve systems of polynomial inequalities. Well, system means more than one equation so that means more graphing................................ But you will not be disappointed because not many people have the opportunity to see me draw pretty pictures.

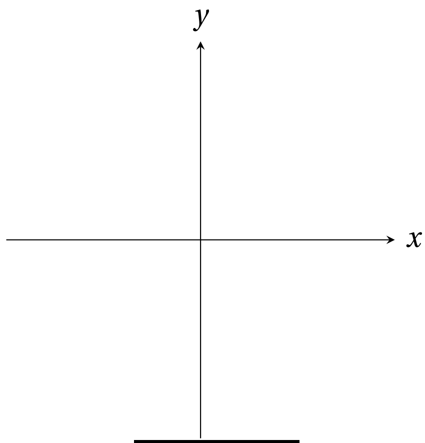
Example 7.11. Let us go back to the very beginning, where life around us is just full of boring lines and elegant parabolas. I want you to sketch the region that satisfies each of the following inequalities.



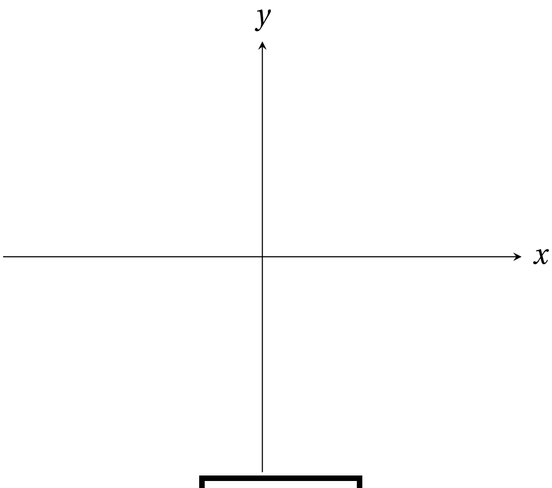
$\boxed{x > -3}$



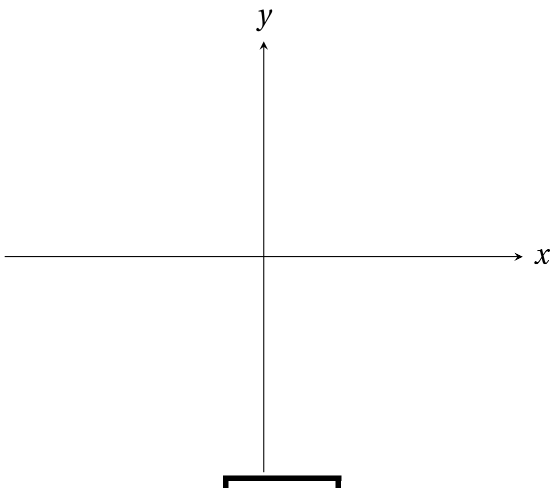
$\boxed{y \leq 5}$



$\boxed{y > 2x - 3}$



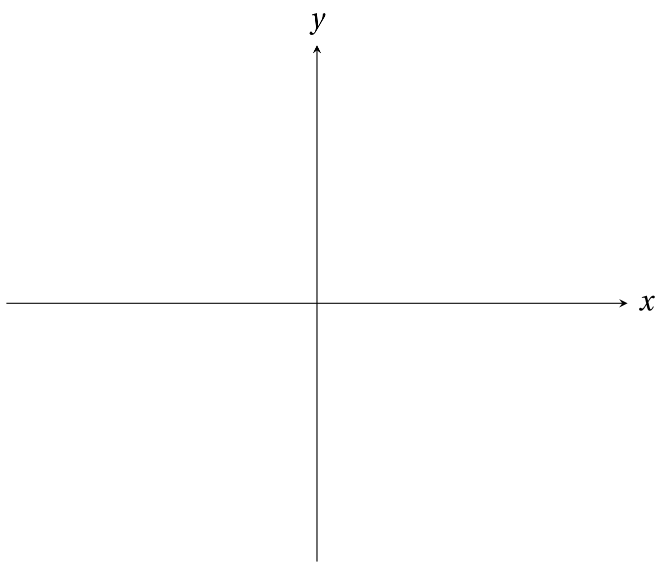
$\boxed{y \geq x^2 + 1}$



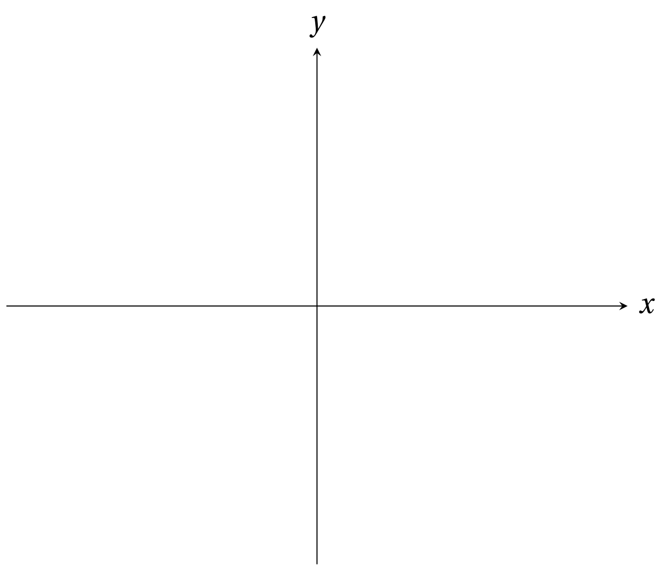
$\boxed{x > y^2}$

Example 7.12 (Lines and parabolas)**.** Let us kick it up a notch. Sketch the solution region for the following systems of inequalities. Carefully label all the vertices of the solution region.

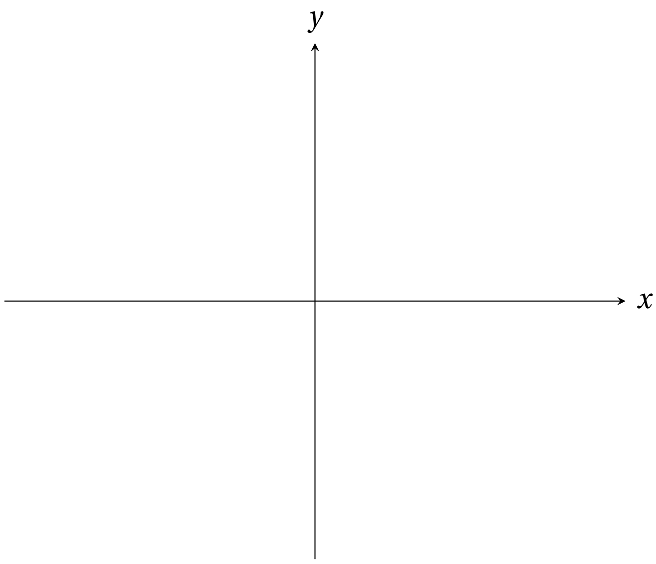
(a) $\begin{cases} y \geq 2x - 3 \\ x \geq -5 \end{cases}$



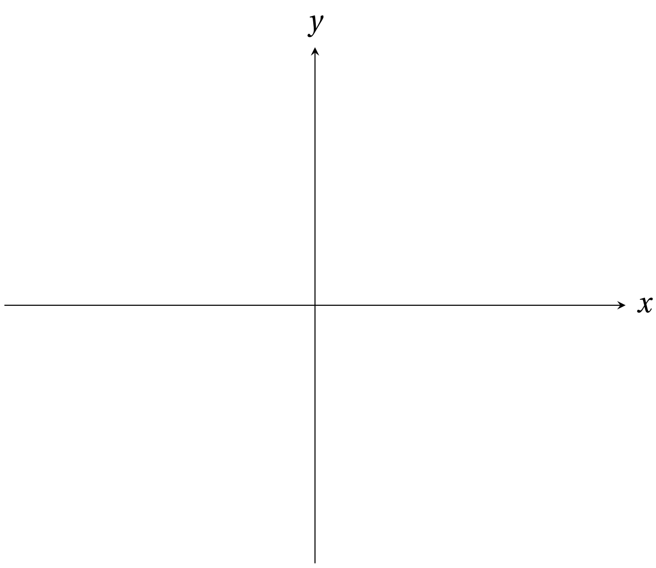
(b) $\begin{cases} 2y + 5x < 2 \\ \qquad\;\; y \geq -10 \end{cases}$



(c) $\begin{cases} y \geq x^2 + 1 \\ y \leq 2 \end{cases}$

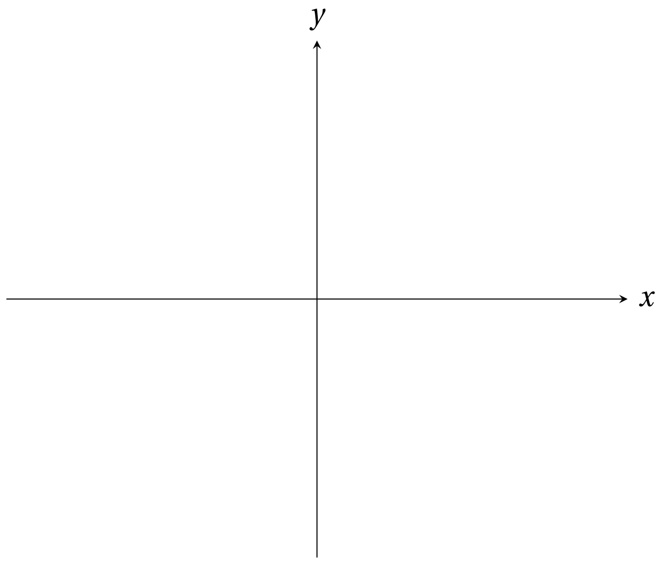


(d) $\begin{cases} x > y^2 \\ x \leq y + 2 \end{cases}$



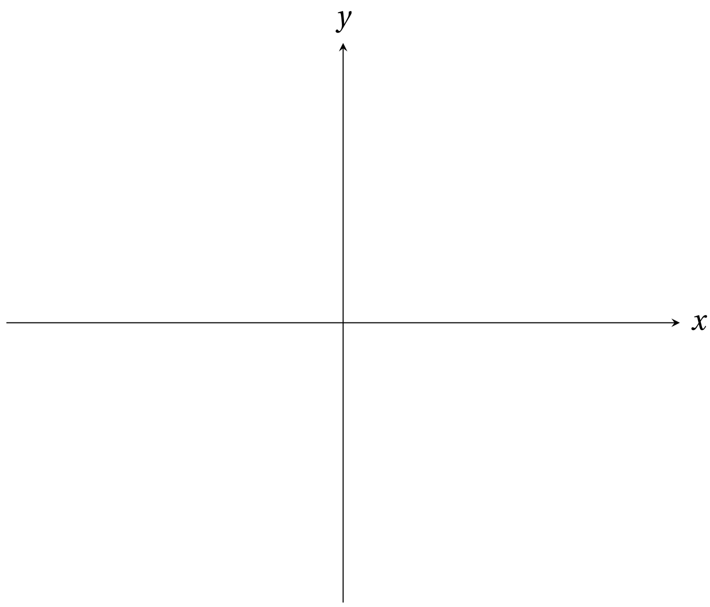
Example 7.13 (A system with no solution)**.** Solve the following system of inequalities.

$$\begin{cases} y < 0 \\ (x-3)^2 + (y-1)^2 \le 9 \end{cases}$$



Example 7.14. Sketch the solution region satisfying the following system of inequalities. Carefully label all the vertices of the solution region.

$$\begin{cases} y \ge 3x \\ x + y \le 2 \\ x \ge -7 \end{cases}$$



8 Matrices and Determinants

8.1 Matrices and Systems of Equations

An $m \times n$ matrix A is a rectangular array of numbers

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}.$$

1. A matrix A is said to be of **order of size** $m \times n$ if it has m rows and n columns.
2. The **entry** a_{ij} of a matrix A is the number in the ith row and the jth column.
3. A matrix that has only one row is called a **row matrix**, and a matrix that has only one column is called a **column matrix**.
4. A matrix A is said to be **square** if the number of rows is the same as the number of columns.
5. The entries $a_{11}, a_{22}, \dots, a_{nn}$ of a square matrix A are called the **diagonal entries** of A.

Example 8.1. Determine the order of the matrix.

(a) $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & -4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & -2 & 3 & 1 \\ 4 & 0 & -6 & 0 \\ 7 & 22 & 13 & 55 \\ 6 & -4 & -1 & 7 \end{bmatrix}$

(c) $A = \begin{bmatrix} -5 & 0 & 3 \end{bmatrix}$

(d) $A = \begin{bmatrix} 8 \\ -1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$

Consider the following 3×3 linear system.

$$\begin{cases} x + 10y - 3z = 2 \\ -5x - 2y + z = -7 \\ 3x - y - 4z = -1 \end{cases}$$

Example 8.2. Write the system of linear equations represented by the augmented matrix.

(a) $\left[\begin{array}{rr:r} 2 & -6 & 15 \\ -3 & 11 & -2 \end{array}\right]$

(b) $\left[\begin{array}{rr:r} 5 & -2 & 1 \\ -1 & 1 & 4 \end{array}\right]$

(c) $\left[\begin{array}{rrr:r} -1 & -4 & 5 & 3 \\ 5 & 0 & 2 & -7 \\ 3 & -6 & -1 & 4 \end{array}\right]$

(d) $\left[\begin{array}{rrr:r} 2 & -1 & 0 & 10 \\ -1 & 0 & 24 & -5 \\ -3 & 7 & -6 & -23 \end{array}\right]$

Remark 8.3. As we previously seen in Section 7.3, given a $n \times n$ linear system for $n \geq 3$, we apply *Gaussian elimination* so that the given linear system reduces to the *staircase form*. We now revisit Gaussian elimination, the only difference is that we do everything in terms of matrix.

Example 8.4. Consider the following system of linear equations.

$$\begin{cases} x + y + z = 6 \\ 2x - 3y + z = -11 \\ -4x + 6y - z = 25 \end{cases}$$

(a) Write the associated coefficient matrix and the augmented matrix.

(b) Solve the system using Gaussian elimination.

Example 8.5. Consider the following system of linear equations.

$$\begin{cases} 3x - 2y + z = 15 \\ -x + y + 2z = -10 \\ x - y - 4z = 14 \end{cases}$$

(a) Write the associated coefficient matrix and the augmented matrix.

(b) Solve the system using Gaussian elimination and back substitution.

There are three possible cases for the solution of the system after we apply Gaussian elimination:

$$\left[\begin{array}{ccc:c} 1 & -2 & 1 & 15 \\ 0 & 1 & 7 & -15 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & -2 & 1 & 15 \\ 0 & 1 & 7 & -15 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & -2 & 1 & 15 \\ 0 & 1 & 7 & -15 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

8.2 Operations with Matrices

Two matrices A and B are said to be equal if they have the same order and same entry. As an example, find x and y given that

$$\begin{bmatrix} 2 & 0 & 3 \\ -1 & 2x-3 & 4 \\ -7 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \\ 1+5y & -5 & 6 \end{bmatrix}.$$

$2x-3 = \qquad \Longrightarrow$

$1+5y = \qquad \Longrightarrow$

Addition and subtraction

We can add or subtract two matrices if and only if they have the same order. To add (subtract) two matrices, simply add (subtract) the corresponding entries. As an example, consider

$$A = \begin{bmatrix} -6 & 3 \\ 8 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ -1 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

$A+B =$

$A-B =$

$C+D =$

$C-D =$

$A-C =$

Scalar multiplication

Scalar multiple of a matrix A by a given real number c simply means multiplying each entry of A by c. Consider

$$A = \begin{bmatrix} 1 & 0 & 7 & 5 \\ -2 & 4 & 1 & -3 \\ 4 & -1 & 0 & 11 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -5 \\ -1 & 4 \end{bmatrix}.$$

$$3A =$$

$$-7B =$$

$$3A - 7B =$$

Example 8.6. Solve for X in the equation $5A = 3X + 2B$, where

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix}.$$

Matrix multiplication

Consider the following system of linear equations.

$$\begin{cases} x + 10y - 3z = 2 \\ -5x - 2y + z = -7 \\ 3x - y - 4z = -1 \end{cases}$$

This system can be written as a **matrix equation** $AX = B$.

To make sense of the matrix product AX, we must **"multiply each row of A with the column matrix X"**. This is exactly how we are going to define matrix multiplication!

Matrix multiplication
$\Updownarrow\Updownarrow\Updownarrow$
Row-by-column multiplication

1. Given two matrices A and B, the **matrix product** AB is defined if and only if the following condition holds:

   $$\text{number of columns of } A = \text{number of rows of } B.$$

2. If A is of order $m \times n$ and B is of order $n \times p$, then the matrix product AB is of order $m \times p$.

3. Unfortunately, **matrix product is not commutative**, *i.e.* the matrix products AB and BA are not the same in general. As an example, consider the following 2×2 matrices

   $$A = \begin{bmatrix} 1 & -3 \\ 5 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 7 & -1 \end{bmatrix}.$$

   We claim that $AB \neq BA$.

   $$AB = \begin{bmatrix} 1 & -3 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 7 & -1 \end{bmatrix} =$$

   $$BA = \begin{bmatrix} 0 & 2 \\ 7 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 5 & -4 \end{bmatrix} =$$

Example 8.7. Find AB, BA, A^2, B^2 if possible.

$$A = \begin{bmatrix} 5 & -3 \\ -1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 3 \\ 1 & -4 \\ 2 & 5 \end{bmatrix}$$

8.3 The Inverse of a Square Matrix

The inverse of a square matrix is similar to that of the inverse of a function. Suppose we want to solve for X in the matrix equation $AX = B$, where A, B are two known matrices. We left-multiply each side of $AX = B$ by some appropriately chosen matrix C that undo the process of multiplying A onto X. More precisely,

$$\begin{aligned} AX &= B \\ CAX &= CB && \left[\text{Left-multiplying each side by } C.\right] \\ I_n X &= CB && \left[C \text{ is chosen so that we have } CA = I_n.\right] \\ X &= CB = A^{-1}B. && \left[I_n X = X \text{ since } I_n \text{ is the identity matrix.}\right] \end{aligned}$$

If we can find such matrix C, then we call it the (multiplicative) **inverse of A** and denote it by A^{-1}, satisfying $A^{-1}A = AA^{-1} = I_n$. **NOT EVERY SQUARE MATRIX HAS AN INVERSE!**

Inverse of a 2×2 matrix

Given a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse A^{-1} is given by

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \text{provided } ad-bc \neq 0.}$$

♣ The quantity $\det(A) = ad - bc$ is called the **determinant of A**.

♣ A has an inverse $A^{-1} \iff \det(A) \neq 0$.

According to the formula, the inverse of a 2×2 matrix A is found as follows:

1.

2. (a)

   (b)

   (c)

Example 8.8. Find the inverse of A if it exists.

(a) $A = \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} -7 & -11 \\ 4 & 8 \end{bmatrix}$

(c) $A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} -8 & -4 \\ 12 & -6 \end{bmatrix}$

(e) $A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$

(f) $A = \begin{bmatrix} 3 & 4 \\ -5 & -3 \end{bmatrix}$

Solving linear system with inverse matrix

Example 8.9. Consider the following system of linear equations.

$$\begin{cases} 3x + y = 5 \\ 7x - 6y = 10 \end{cases}$$

(a) Write the corresponding matrix equation.

(b) Find the inverse of the coefficient matrix.

(c) Use the inverse matrix to solve the system.

Example 8.10. Consider the following system of linear equations.

$$\begin{cases} 5x - y = -11 \\ -3x + 2y = 4 \end{cases}$$

(a) Write the corresponding matrix equation.

(b) Find the inverse of the coefficient matrix.

(c) Use the inverse of the coefficient matrix to solve the system.

Example 8.11. Consider the system of linear equations.

$$\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1 \\ 2x + 4y + z = -2 \end{cases}$$

(a) Write the corresponding matrix equation.

(b) Use the fact that the inverse of the coefficient matrix is $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$ to solve the system.

9 Sequences and Series

9.1 Sequences and Series

A **sequence**, denoted by (a_n), is an ordered list of numbers

$$(a_n) = a_1, a_2, a_3, a_4, a_5, \dots\dots\dots$$

1. If n stops at a certain fixed number, then (a_n) is a **finite sequence**. Otherwise it is an **infinite sequence**, usually indicated by the dots $\dots\dots\dots$ as seen above.

2. The $a_1, a_2, a_3, \dots$ are the **terms** of the sequence.

3. We usually choose n starting from 1, otherwise it is always indicated in the given sequence where n starts.

4. Any ordered list of numbers are sequences, in particular they need not satisfy any sort of patterns.

Writing the terms of a sequence

Example 9.1. Write the first four terms of the following sequences.

1. $a_n = 3n - 5$, $n \geq 1$

2. $a_n = \dfrac{2n}{(n^2+1)^2}$, $n \geq 1$

3. **Alternating sequence**, where the terms alternate in sign: $a_n = \dfrac{(-1)^n}{4n+1}$, $n \geq 1$

4. **Recursive sequence**, where the terms are defined using previous terms:

$$a_1 = 7, \;\; a_n = a_{n-1} - 3, \;\; n \geq 2.$$

Example 9.2. Given the sequence $a_n = \dfrac{4+(-1)^n}{(3n-1)}$, $n \geq 1$, find a_3, a_7 and a_{12}.

Finding the nth term of a sequence

Assuming n begins with 1, write an expression for the apparent nth term (a_n) of each sequence.

1. $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, \ldots\ldots$

2. $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \frac{10}{11}, \ldots\ldots$

3. $-7, -1, 5, 11, 17, \ldots\ldots$

4. $3, 8, 15, 24, 35 \ldots\ldots$

Summation notation and series

Definition 9.3.

1. The sum of the first n terms of a sequence is called a **finite series** or **nth partial sum** of the sequence. It is denoted by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n, \quad \text{where } i = \text{index of summation,}$$
$$1 = \text{lower limit of summation,}$$
$$n = \text{upper limit of summation.}$$

   We read this as "sum over i from 1 to n of a_i".

2. The sum of all the terms of an infinite sequence is called an **infinite series**. It is denoted by

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \dots\dots\dots$$

Example 9.4. Compute the following sum.

1. $\sum_{i=1}^{6} (3 - 4i)$

2. $\sum_{j=0}^{4} (-2)^{j+1}$

3. $\sum_{k=1}^{5} \left(1 + k^3\right)$

9.2 Arithmetic Sequences and Partial Sums

Definition 9.5. An **arithmetic sequence** $(a_n) = a_1, a_2, a_3, \ldots\ldots$ is a sequence where the **differences** between two consecutive (successive) terms are the same, *i.e.*

$$d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = a_5 - a_4 = \ldots\ldots$$

The number d is called the **common difference** of the arithmetic sequence.

Example 9.6. Determine if each of the following sequences is an arithmetic sequence. If so, find the common difference d.

1. $(a_n) = 1, 4, 9, 16, \ldots\ldots$

2. $(a_n) = 3, 8, 13, 18, 23, \ldots\ldots$

3. $(a_n) = 7, 3, -1, -5, -9, \ldots\ldots$

Example 9.7. Write the first 4 terms and find a formula a_n for the following arithmetic sequences.

1. $a_1 = 3$ and $d = 2$

2. $a_1 = 15$ and $d = -7$

Example 9.8. Find the common difference and a_{150} of the following arithmetic sequences.

1. $a_n = 6 - 4n$, $n \geq 1$

2. $a_n = 7n + 2$, $n \geq 1$

Example 9.9. Consider the arithmetic sequence $3, 7, 11, 15, \ldots\ldots$. Write down a formula (a_n) for the given sequence and find a_{101}.

The **sum of a finite arithmetic sequence** with n terms is given by

$$S_n = \frac{n}{2}\left[a_1 + a_n\right], \quad \text{where } a_1 = \text{the first term we add}$$
$$a_n = \text{the last term we add}$$
$$n = \text{number of terms we are summing.}$$

Example 9.10.

(a) Find the 20th partial sum of the arithmetic sequence $-8, -2, 4, 10, 16, \ldots\ldots$

(b) Given the arithmetic sequence $35, 26, 17, 8, -1, \ldots\ldots$, find S_{100}.

Example 9.11. Compute the following sums.

1. $\sum_{i=1}^{100} (7i - 2)$

2. $\sum_{j=1}^{50} \left(3 + 4j\right)$

3. $\sum_{k=1}^{600} (2k + 25)$

9.3 Geometric Sequences and Series

Definition 9.12. A **geometric sequence** $(a_n) = a_1, a_2, a_3, \ldots\ldots$ is a sequence where the **ratios** of two consecutive (succesive) terms are the same, *i.e.*

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \frac{a_5}{a_4} = \ldots\ldots.$$

The number r is called the **common ratio** of the geometric sequence.

Example 9.13. Determine if each of the following sequences is a geometric sequence. If so, find the common ratio r.

1. $(a_n) = 12, 36, 108, 324, \ldots\ldots$

2. $(a_n) = 3, 5, 7, 9, 11, \ldots\ldots$

3. $(a_n) = -\frac{5}{2}, \frac{5}{6}, -\frac{5}{18}, \frac{5}{54}, \ldots\ldots$

Example 9.14. Write the first 4 terms and find a formula a_n for the following geometric sequences.

1. $a_1 = 11$ and $r = 3$; find a_{50}.

2. $a_1 = -7$ and $r = \frac{1}{4}$; find a_{123}.

3. $a_1 = 2$ and $r = -5$; find a_{28}.

Example 9.15. Consider the geometric sequence $7, 21, 63, 189, \ldots\ldots$ Write down a formula (a_n) for the given sequence and find the 50th term.

The sum of all the terms of a given infinite geometric sequence is called an **infinite geometric series**, given by

$$S = \sum_{n=0}^{\infty} a_0 r^n = \sum_{n=1}^{\infty} a_1 r^{n-1} = \begin{cases} \dfrac{a_{\text{first}}}{1-r} & \text{if } |r| < 1, \\ \text{does not exists (DNE)} & \text{if } |r| \geq 1. \end{cases}$$

Example 9.16. Find the sum of the following infinite geometric series.

1. $\displaystyle\sum_{n=0}^{\infty} 4\left(\frac{1}{6}\right)^n$

2. $\displaystyle\sum_{n=0}^{\infty} -3(0.6)^n$

3. $\displaystyle\sum_{n=1}^{\infty} 11\left(-\frac{5}{7}\right)^{n-1}$

Example 9.17. Find the sum of the following infinite geometric series if possible.

(a) $-3, \frac{3}{2}, -\frac{3}{4}, \frac{3}{8}, \ldots\ldots$

(b) $\frac{1}{3}, \frac{5}{3}, \frac{25}{3}, \frac{125}{3}, \ldots\ldots$

(c) $\frac{7}{3}, \frac{7}{9}, \frac{7}{27}, \frac{7}{81}, \ldots\ldots$